ROLLE LEAVES AND O-MINIMAL STRUCTURES

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By

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Abstract

Let $\overline{\mathcal{R}}$ be a real closed field, and let $\mathcal{R}$ be a (model-theoretic) expansion of $\overline{\mathcal{R}}$ with the intermediate value property (IVP). We develop a version of Khovanskii theory relative to $\mathcal{R}$ over an o-minimal expansion of $\overline{\mathcal{R}}$. We also introduce a notion of the relative Pfaffian closure $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ of an o-minimal structure $\widetilde{\mathcal{R}}$ with respect to an expansion $\mathcal{R}$ that has the IVP. We prove that $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ is o-minimal when $\mathcal{R}$ is a model of the real projective hierarchy. Using this result, we obtain a strong uniformity result on definable sets in $\mathscr{P}(\widetilde{\mathbb{R}})$, the Pfaffian closure of an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field.

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Contents

1 Introduction

Model theory has the ability to shed light on diverse branches of mathematics. Consider this Proposition due to Van den Dries [27]:

Proposition 1.1. *For any natural numbers m and n, there are only finitely many homeomorphism types among the sets*

$$Z(f) := \{x \in \mathbb{R}^n : f(x) = 0\},$$

where $f(x_1, ..., x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ has at most m monomials.

While one need not know any model theory to appreciate its statement, the proof of this result was first noticed as a consequence of the following celebrated Theorem of Wilkie [29]. Let $\overline{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$ be the field of real numbers, and let $\mathbb{R}_{\exp} := \langle \overline{\mathbb{R}}, \exp \rangle$ be its expansion by the real exponential function.

Theorem 1.2. *The structure $\mathbb{R}_{\exp}$ is o-minimal.*

This deceptively brief statement, combined with the powerful theory of o-minimal structures, implies that every subset of $\mathbb{R}^n$ that is defined with exponential polynomials can be partitioned into finitely many simple sets called cells. (See Definition 4.2 below.)

Any mathematician may sense that the sets $Z(f)$ should not vary too wildly; it is the machinery of logic—forged at the foundations of mathematics—that makes this intuition precise: The collection of all sets $Z(f)$ with f having at most m monomials is contained in the union of finitely many definable families in $\mathbb{R}_{\exp}$. Thus all of the tools of o-minimality apply.

The study of exponentially definable sets was pioneered, under the heading "fewnomials," by Khovanskii [14], whose work plays a central role in Wilkie's proof. These investigations drew attention to the class of Pfaffian functions, which we now define.

Definition 1.3. A C^1-smooth function $f_m : \mathbb{R}^n \to \mathbb{R}$ is **Pfaffian** if there are C^1-functions $f_1, \ldots, f_{m-1} : \mathbb{R}^n \to \mathbb{R}$ and polynomials $P_{i,j}(x_1, \ldots, x_n, y_1, \ldots, y_i)$ for $1 \le i \le m$ and $1 \le j \le n$ such that

$$\frac{\partial f_i}{\partial x_j} = P_{i,j}(x_1, \ldots, x_n, f_1, \ldots, f_i) \tag{1.1}$$

for each i, j.

Notice that the exponential function is Pfaffian; the sine function is not. It was inferred that the "triangular" character of the system (1.1) should prohibit solutions from exhibiting infinite oscillatory behavior. This insight is witnessed by another Theorem of Wilkie [30]:

Theorem 1.4. *The structure* $\mathbb{R}_{\mathrm{Pfaff}} := \langle \overline{\mathbb{R}}, \mathcal{F} \rangle$ *is o-minimal, where* $\mathcal{F}$ *is the collection of all Pfaffian functions.*

Further generalizations of this Theorem have subsequently been produced by Lion and Rolin [18] , Karpinski and Macintyre [12], and Speissegger [25], the latter being the most general. To state this result, we recall some definitions.

Definition 1.5. Let ω be a nonsingular 1-form of class C^1 on an open subset U of $\mathbb{R}^n$. An **integral manifold of** $\omega = 0$ is a manifold M of dimension $n-1$ and class C^1 such that $M \subset U$ and $T_aM = \ker(\omega(a))$ for all $a \in M$. A **leaf** L of $\omega = 0$ is a maximal connected embedded integral manifold of $\omega = 0$. A leaf L of $\omega = 0$ is called **Rolle** if L is relatively closed in U and satisfies the following additional property:

> [Rolle Property] For each C^1 curve $\gamma : [0,1] \to U$ with $\gamma(0) \in L$ and $\gamma(1) \in L$, there is a $t \in [0,1]$ such that $\omega(\gamma(t))\gamma'(t) = 0$.

Remarks: The term "Rolle" refers to Rolle's Theorem from classical analysis, which can be expressed in the following way: The x-axis is a Rolle leaf of the equation $dy = 0$. Rolle leaves were introduced by Moussu and Roche [21].

The main result of [25] can now be stated thus:

Theorem 1.6. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of $\overline{\mathbb{R}}$. Then there is an o-minimal structure $\mathscr{P}(\widetilde{\mathbb{R}})$ satisfying the following condition: Whenever ω is a nonsingular 1-form of class C^1 that is definable in $\mathscr{P}(\widetilde{\mathbb{R}})$ on an open subset U of $\mathbb{R}^n$ and whenever a set L is a Rolle leaf of $\omega = 0$, then L is definable in $\mathscr{P}(\widetilde{\mathbb{R}})$.*

The structure $\mathscr{P}(\widetilde{\mathbb{R}})$ is called the **Pfaffian Closure** of $\widetilde{\mathbb{R}}$ since it can be shown that every Pfaffian function is definable in $\mathscr{P}(\widetilde{\mathbb{R}})$.

The central objective of this dissertation is to generalize Theorem 1.6 and its proof. Specifically, our goal is to produce a version that does not require the underlying universe of the structure to be $\mathbb{R}$. Put in yet another way, when $\widetilde{\mathcal{R}}$ is any o-minimal expansion of a real closed field $\overline{\mathcal{R}}$, we investigate whether $\widetilde{\mathcal{R}}$ has an o-minimal Pfaffian closure of some kind.

For fundamental reasons, the naïve generalization of Theorem 1.6 is hopeless: Kuhlmann and Shelah show in [16] that for each regular uncountable cardinal κ there is a real closed field of cardinality κ that admits 2^κ pairwise nonisomorphic models of $\mathrm{Th}(\mathbb{R}_{\exp})$. Thus there are 2^κ distinct exponential functions on some real closed field R. Each of these functions is a solution to the initial value problem $y' = y$, $y(0) = 1$; hence each should be Pfaffian in a generalized sense. On the other hand, Proposition 2.11 below implies that no o-minimal structure can define more than one such solution. Consequently, any notion of Pfaffian closure that is expected to preserve o-minimality must carry additional restrictions.

A first restriction suggested by the preceding discussion is to fix as a workspace an expansion $\mathcal{R}$ of $\widetilde{\mathcal{R}}$ that has the Intermediate Value Property (IVP for short) as given by Definition 2.1. Then we consider only Rolle $\mathcal{R}$-leaves (Definition 5.2), which are required to be definable in $\mathcal{R}$. While it is not yet settled whether this condition is sufficient, we do obtain in this setting a relativised version of the Khovanskii theory from [25] (Theorem 5.1 below). We prove the following in

Chapter 5:

Theorem 1.7. *Let $\widetilde{\mathcal{R}}$ be an o-minimal expansion of a real closed field $\overline{\mathcal{R}}$, and let $\mathcal{R}$ be an expansion of $\widetilde{\mathcal{R}}$ that has the* IVP*. Let $\Omega = (\omega_1, \dots, \omega_l)$ be a tuple of $\widetilde{\mathcal{R}}$-definable nonsingular 1-forms defined on a common open subset U of R^n, and let A be a an $\widetilde{\mathcal{R}}$-definable subset of U. Then there is a natural number K such that whenever L_i is a Rolle $\mathcal{R}$-leaf of $\omega_i = 0$ for each $i = 1, \dots, l$, the set $A \cap L_1 \cap \dots \cap L_l$ is a union of fewer than K definably connected $\mathcal{R}$-manifolds. (See Definition 3.1.)*

Unfortunately, the proof of Theorem 1.6 makes casual use of additional results that are harder to generalize—the existence of Hausdorff limits and the Baire category theorem are two such results. To obtain versions of these, we need to restrict ourselves further: Let $\mathbb{R}_{\mathrm{proj}}$ denote $\langle \overline{\mathbb{R}}, \mathbb{Z} \rangle$, the expansion of $\overline{\mathbb{R}}$ by a predicate for the integers. Let T^{proj} be the first-order theory $\mathrm{Th}(\mathbb{R}_{\mathrm{proj}})$ of $\mathbb{R}_{\mathrm{proj}}$. The following Theorem is our main result. (Its proof fills Chapters 6-8.)

Theorem 1.8. *Suppose $\mathcal{R}$ is a model of T^{proj}, and let $\widetilde{\mathcal{R}}$ be an o-minimal reduct of $\mathcal{R}$ that still expands the field $\overline{R}$. Then there is an o-minimal expansion $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ of $\widetilde{\mathcal{R}}$ with the following property: Whenever ω is a nonsingular* 1*-form of class C^1 on an open subset U of R^n that is definable in $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ and whenever $L \subset U$ is a Rolle $\mathcal{R}$-leaf of $\omega = 0$, then L is definable in $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$.*

Note: We use the terms “expansion” and “reduct” in the sense of definability—that is, our hypothesis is that every $\widetilde{\mathcal{R}}$-definable set is definable in $\mathcal{R}$ and that every $\overline{\mathcal{R}}$-definable set is definable in $\widetilde{\mathcal{R}}$. Thus Theorem 1.8 indeed generalizes Theorem 1.6, as $\mathscr{P}(\widetilde{\mathbb{R}}, \mathbb{R}_{\mathrm{proj}})$ is identical to the structure $\mathscr{P}(\widetilde{\mathbb{R}})$.

While the assumption that $\mathcal{R}$ is a model of T^{proj} is extremely strong—some consequences are surveyed in Chapter 6—Theorem 1.8 permits us for the first time to apply model-theoretic compactness arguments to the study of $\mathscr{P}(\widetilde{\mathbb{R}})$. We illustrate this next with a result in the spirit of Proposition 1.1.

Let $\widetilde{\mathscr{L}}$ be the language of an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$. It is convenient to identify a nonsingular 1-form $\omega = a_1(x)dx_1 + \cdots + a_n(x)dx_n$ on a subset of $\mathbb{R}^n$ with the nonvanishing vector field $F_\omega = (a_1(x), \ldots, a_n(x))$ of its component functions. Then we say that a manifold L is a **Rolle leaf of** F_ω if L is a Rolle leaf of $\omega = 0$.

Let $\mathcal{P}$ be a collection of predicate symbols P that are not contained in $\widetilde{\mathscr{L}}$. Suppose ϕ is a formula in the language $\widetilde{\mathscr{L}} \cup \{P_1, \ldots, P_j\}$ such that each P_i is in $\mathcal{P}$ and has arity n_i. Let $\chi_1, \ldots, \chi_j$ be formulas in an expansion $\mathscr{L}$ of $\widetilde{\mathscr{L}}$ such that each χ_i has free variables $x^i := (x_1, ..., x_{n_i})$. By convention we write $\phi[\chi_1, \ldots, \chi_j]$ to denote the $\mathscr{L}$-formula obtained from ϕ by replacing each occurrence of $P_i(x^i)$ with the corresponding formula χ_i.

We need one more definition. Take $\widetilde{\mathscr{L}}$ to be the language of $\mathscr{P}(\widetilde{\mathbb{R}})$, and let $\Phi := (\phi_0, \ldots, \phi_j)$ be a finite tuple of $(\widetilde{\mathscr{L}} \cup \mathcal{P})$-formulas.

Definition 1.9. Let X be a subset of $\mathbb{R}^n$ that is definable in $\mathscr{P}(\widetilde{\mathbb{R}})$. We say X **has format** Φ if there are formulas χ_i for $i = 1, \ldots, j$ such that the following hold:

(1) Each ϕ_i is in the language $\widetilde{\mathscr{L}} \cup \{P_1, \ldots, P_i\}$, and each χ_i is in some expansion $\mathscr{L}$ of $\widetilde{\mathscr{L}}$.

(2) For $i = 0, \ldots, j-1$, each $\phi_i[\chi_1, \ldots, \chi_i]$ defines the graph of a nonvanishing vector field on an open set U_i.

(3) For $i = 0, \ldots, j-1$, each χ_{i+1} defines a Rolle leaf of the vector field defined by $\phi_i[\chi_1, \ldots, \chi_i]$.

(4) The set X is defined by the formula $\phi_j[\chi_1, \ldots, \chi_j]$.

With this terminology, we derive (in Chapter 10) the following consequence from Theorem 1.8:

Theorem 1.10. *For each Φ as above, there is a natural number K such that whenever a set X has format Φ, the set X has fewer than K connected components.*

This Theorem is another generalization of the Khovanskii Theory from [25]. The definition of the "format" of a set was inspired by Gabrielov's work in [7]. There he defines the format of a "limit set" in order to derive an effective bound on the number of its components. In contrast, the bounds given by Theorem 1.10 are not effective; we know of no way to compute them. This suggests the following question:

> Suppose the language $\widetilde{\mathscr{L}}$ is countable. Is there an algorithm that, given a format Φ, provides a bound K on the number of connected components of sets X with format Φ?

Additional open questions appear in Chapter 11, as well as other implications of this research.

We have made an effort to keep our exposition reasonably self-contained; we state and provide references for all those results that we use without proof. Lest any doubts remain at the foundation of our development, we take pains to verify expected analogues of results from advanced calculus and elementary differential topology. These sections were written with the ulterior motive of documenting folklore and expectation.

In contrast, we require the reader to have some background in general model theory, such as can be found in Chang and Keisler [2] or Hodges [10]. Throughout this dissertation, we shall make claims that take for granted a familiarity with the expressive power and mechanics of first-order formulas.

Conventions: We fix a structure $\mathcal{R}$ that expands a real closed ordered field

$$\overline{\mathcal{R}} := \langle R, +, \cdot, 0, 1, < \rangle.$$

(For background on real closed fields see Bochnak et al. [1] for the geometric approach and Marker [19] for a model-theoretic treatment.) Unless explicitly stated otherwise, the term "definable" will mean "definable with parameters."

The letters i, j, k, l, m, n, p, q range over natural numbers, and the letters r, s, t range over R. Unless stated otherwise, the letters x, y, and z denote tuples of variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_l)$ that range over R^n, R^m, and R^l respectively.

Given vectors $v_1, \ldots, v_k$ in R^n, we write $\langle v_1, \ldots, v_k \rangle$ for the R-linear span of the set $\{v_1, \ldots, v_k\}$. We also let $\{e_1, \ldots, e_n\}$ be the standard basis for R^n. For a single vector v in R^n, we let $v^\perp := \{w \in R^n : v \cdot w = 0\}$. Similarly, for a vector subspace V of R^n, we put $V^\perp := \{w \in R^n : v \cdot w = 0 \text{ for all } v \in V\}$.

Given an r in R, we set $|r| := \max\{r, -r\}$ and equip R^n with the distance function

$$d(x, x') := \max\{|x_1 - x'_1|, \ldots, |x_n - x'_n|\}.$$

As usual this yields a metric on R^n whose induced topology agrees with the order topology on R and the corresponding product topology on R^n. (Note: our metrics take values in R rather than $\mathbb{R}$.)

For a in R^n, we use the notation $||a||$ to represent $d(a, 0)$; and for positive t in R, we let $B(a, t)$ denote $\{x \in R^n : d(x, a) < t\}$, the open ball of radius t around a.

Let A be a subset of R^n. We use $|A|$ to denote the cardinality of A. We write $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(A)$ and $\mathrm{fr}(A) := \mathrm{cl}(A) \setminus A$ for the topological closure, interior, boundary, and frontier of A respectively. We also define the sets

$$T(A, t) := \{x \in R^n : d(x, a) < t \text{ for some } a \in A\},$$

$$S(A, t) := \{x \in R^n : d(x, a) \leq t \text{ for some } a \in A\}.$$

For a function $f : A \to R$, we denote the graph of f by

$$\Gamma(f) := \{(x,t) \in A \times R : f(x) = t\}.$$

Similarly, we define the sets

$$(f, \infty) := \{(x,t) \in A \times R : f(x) < t\},$$

$$(-\infty, f) := \{(x,t) \in A \times R : t < f(x)\}.$$

If $g : A \to R$ is another function and $f(x) < g(x)$ for all $x \in A$, then we define

$$(f, g) := \{(x,t) \in A \times R : f(x) < t < g(x)\}.$$

If C is a subset of A, then $f|_C$ denotes the restriction of f to C.

For $B \subseteq R^{n+m}$, we let B_x denote the fiber $\{y \in R^m : (x,y) \in B\}$ of B over x. We also write $\mathrm{cl}(B)_x$ for $(\mathrm{cl}(B))_x$, which is not to be confused with $\mathrm{cl}(B_x)$.

For a strictly increasing map $\iota : \{1, \ldots, k\} \to \{1, \ldots, n\}$, we let $\Pi_\iota : R^n \to R^k$ be the projection given by $\Pi_\iota(x_1, \ldots, x_n) := (x_{\iota(1)}, \ldots, x_{\iota(k)})$. If $\iota(i) = i$ for $i = 1, \ldots, k$, we also write Π_k in place of Π_ι.

We now begin our journey.

2 Preliminaries on the IVP

It is easy to prove that an ordered field that is connected in its order topology is isomorphic to $\overline{\mathbb{R}}$. (A connected ordered field is archimedian, and hence embeds into the reals; by connectedness again, this embedding must be surjective.) Consequently, connectedness is not a first-order property. In [20], Miller suggests an alternative called the **Intermediate Value Property** (IVP for short) that is first-order and successfully replaces connectedness in many arguments. This chapter reviews and records some consequences and equivalents of the IVP. First, the definitions:

Definition 2.1. The structure $\mathcal{R}$ has the IVP if for all a and b in R, each continuous definable function $f : [a, b] \to R$ takes on every value in R between $f(a)$ and $f(b)$.

Definition 2.2. A subset A of R^n is **definably connected** (with respect to $\mathcal{R}$) if for each pair of disjoint open definable subsets U and V of R^n such that $A = (A \cap U) \cup (A \cap V)$, we have either $A \cap U = \emptyset$ or $A \cap V = \emptyset$.

(Note: In the definition given in [20], the set A is required to be definable; here it is not.)

These notions are closely related as illustrated by the next Proposition from [20].

Proposition 2.3. *The following are equivalent:*

(i) $\mathcal{R}$ *has the* IVP*.*

(ii) R *is definably connected.*

(iii) Intervals in R are definably connected.

(iv) If A is a nonempty definable subset of R that is bounded above (resp. below), then $\sup A$ *(resp.* $\inf A$*) exists in R.*

(v) If $f : A \to R^n$ is a definable continuous function and A is a closed and bounded subset of R^m, then the set $f(A)$ is closed and bounded.

(vi) If $f : A \to R$ is a definable continuous function and A is a closed and bounded subset of R^n, then f attains a max and min in A.

Observe that every expansion of $\langle \mathbb{R}, < \rangle$ has the IVP. There are also many ways to see that if $\mathcal{R}$ has the IVP, then any model of $T := \mathrm{Th}(\mathcal{R})$ also has the IVP. In this case, we also say that the theory T has the IVP.

Proof. The proof in [20] shows (i)⇔(ii)⇔(iii)⇔(iv)⇒(v)⇒(vi). Here is the remaining implication, (vi)⇒(iv): Let A be a nonempty definable subset of R that is bounded above, and assume for a contradiction that $\sup A$ does not exist. Define

$$\widetilde{A} = \{t \in R : \exists s \in A, t < s\},$$

so that $\widetilde{A}$ is a definable initial segment of R. Choose $a \in \widetilde{A}$ and $b \in R \backslash \widetilde{A}$ and define $f : [a, b] \to R$ by

$$f(t) = \begin{cases} t & \text{if } t \in \widetilde{A} \\ a & \text{if } t \notin \widetilde{A} \end{cases}$$

Since $\sup A$ does not exist, the function f is continuous. By (vi), $\max f = f(c)$ for some $c \in [a, b]$. There are two cases. If $c \in \widetilde{A}$ then, since c is not the supremum of A, there is a $d \in \widetilde{A}$ such that $c < d$. This means that $f(c) = c < d = f(d)$, which cannot be. Suppose on the other hand that $c \notin \widetilde{A}$. If we choose any $d \in \widetilde{A} \backslash \{a\}$, then $f(c) = a < d = f(d)$—a contradiction. □

We assume henceforth that $\mathcal{R}$ has the IVP, and we use Proposition 2.3 without further mention. In this context, we introduce another piece of notation: For a definable subset B of R^n, we define the distance $d(a, B)$ from a to the set B by

$$d(a, B) := \inf\{d(a, b) : b \in B\}. \tag{2.1}$$

Note that for a fixed set B, the function $d(-, B) : R^n \to R$ is continuous. With this in hand, we confirm an expected property:

Lemma 2.4. *Let A be a definable and definably connected subset of R^n. If B is a definable subset of A that is both relatively open and closed in A, then either $B = A$ or $B = \emptyset$.*

Remark: Though this Lemma seems obvious, it is not clear that it holds in ordered structures that do not expand a densely ordered abelian group. It also shows that the (a priori different) definition of "definably connected" given in [27] agrees in our setting with the definition above.

Proof. Let $C = A \backslash B$. Suppose for a contradiction that $B \neq \emptyset$ and $C \neq \emptyset$. We need to find disjoint open definable subsets U and V of R^n such that $B = A \cap U$ and $C = A \cap V$. If a is in A, then at least one of the distances $d(a, B)$ or $d(a, C)$ is zero. On the other hand, if both $d(a, B)$ and $d(a, C)$ are zero, then $a \in \mathrm{cl}(B) \cap \mathrm{cl}(C)$ contradicting that $B \cap C = \emptyset$. Consequently, the two disjoint open sets

$$U := \{a \in R^n : d(a, B) < d(a, C)\}$$

$$V := \{a \in R^n : d(a, C) < d(a, B)\}$$

satisfy our requirements. $\square$

Definition 2.5. Let A be a subset of R^n. Then a subset C of A is a **definably connected component of** A (or just a **component of** A for short) if C is a maximal definably connected subset of A.

Definably connected components always exist by Zorn's lemma, and distinct components of a given set are disjoint. Though they need not be definable, we do have the following:

Proposition 2.6. *If A is a definable set with only finitely many components $C_1, \ldots, C_l$, then each C_i is definable and both open and closed in A.*

Proof. It suffices to show that C_1 is definable and open. For each $i \neq 1$, the union $C_1 \cup C_i$ is not definably connected. Thus there are disjoint open definable sets U_i and V_i such that $C_1 \cup C_i \subset U_i \cup V_i$, and both $(C_1 \cup C_i) \cap U_i$ and $(C_1 \cup C_i) \cap V_i$ are nonempty. By definable connectedness, each of C_1 and C_i meet exactly one of U_i and V_i. At the same time, neither U_i nor V_i can contain both C_1 and C_i. Thus after interchanging the sets U_i and V_i if necessary, we may assume that $C_1 \subset U_i$ and $C_i \subset V_i$. Then since $C_1 = A \cap \bigcap_{i=1}^{l} U_i$, it is definable and open in A. □

Before moving on, we state two more facts we shall use frequently without mention. Their proofs are standard.

Fact 2.7. *Let C be a definably connected subset of a definable set A, and let $f : A \to B$ be a definable continuous function. Then the image $f(C)$ of C is also definably connected. More generally, if C is a set with fewer than K components, then $f(C)$ has fewer than K components.*

Fact 2.8. *Suppose A is a subset of B. Then each component of A is contained in a component of B.*

In Chapter 7 of [27], the definition of a derivative is extended to arbitrary ordered fields, and standard elementary properties are verified. We restate these definitions for convenience and to set our notation.

Definition 2.9. Let U be an open subset of R^n, let a be in U, and let $f : U \to R$ be a definable function. We define $D_i f(a)$ the **(1-st order) partial derivative**

of f **at** a **with respect to** x_i to be the limit

$$D_i f(a) := \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

if this limit exists. The function f is said to be **of class** C^1 **on** U if the partial derivatives $D_1 f, \ldots, D_n f$ exist at each $a \in U$ and are continuous as functions on U. When $n = 1$, we also write f' for $D_1 f$. We recursively define a $(k+1)$**-st order partial derivative of** f to be one of the functions $D_1 g, \ldots, D_n g$ where g is a k-th order partial derivative. If each k-th order partial derivative of f exists and is continuous on U, we say f is of **class** C^k **on** U.

Similarly if $f = (f_1, \ldots, f_m) : U \to R^m$ is a definable function, then f is of **class** C^k on U if the component functions f_j are of class C^k for $j = 1, \ldots, m$. If f is of class C^1, we define the **differential** $d_a f$ **of** f **at** a to be the $(m \times n)$-matrix of partial derivatives $(D_i f_j(a))$, and the **rank of** f **at** a to be the rank of this matrix. When $m = 1$ we also write $\nabla f(a)$ for $d_a f$.

Suppose U and V are open subsets of R^n and $\sigma : U \to V$ is a definable bijection. Then σ is called a **diffeomorphism of class** C^k if both σ and its inverse are of class C^k. The function σ is simply called a **diffeomorphism** if it is a diffeomorphism of class at least C^1.

All of the usual elementary properties of derivatives and differentials carry over to this setting—for instance, given definable functions $g : U \to V$ and $f : V \to W$ of class C^1 we have the chain rule:

$$d_a(f \circ g) = (d_{g(a)} f)(d_a g)$$

where the multiplication on the right is that of matrices.

Following these definitions, [27] goes on to prove versions of several classical results in the o-minimal setting. Replacing o-minimality with the IVP, many of those proofs still work. For example, the proof of the Mean Value Theorem goes through verbatim:

Theorem 2.10 (Mean Value Theorem). *Suppose* $a < b$ *in* R *and that* $f : [a, b] \to R$ *is a definable function. If the derivative* $f'(t)$ *exists for all* $t \in [a, b]$*, then for some* $c \in (a, b)$*, we have* $f(b) - f(a) = (b - a)f'(c)$*.*

Using this, we get uniqueness for definable solutions of certain differential equations.

Proposition 2.11. *Let* I *and* J *be open intervals in* R*. Suppose* $f, g : I \to J$ *are definable functions of class* C^1 *on* I *such that the set* $\{t \in I : f(t) = g(t)\}$ *is nonempty. Suppose also that* $F : I \times J \to R$ *is a definable function and that the partial derivative* D_2F *exists and is continuous on* $I \times J$*. Suppose finally that for all* t *in* I *it is the case that* $f'(t) = F(t, f(t))$ *and* $g'(t) = F(t, g(t))$*. Then* $f = g$*.*

As a special case, this implies that an expansion of a real closed field with the IVP cannot define two distinct exponential functions. To prove the Proposition we need a Lemma.

Lemma 2.12. *Suppose* I *is an open interval in* R*. Suppose* $u : I \to R$ *is a definable function, that* $u'(t)$ *exists for all* t *in* I*, and that* $u(t) = 0$ *for some* t *in* I*. Suppose finally that for each* t_0 *in* I *there is a neighborhood* V *of* t_0 *and an* r *in* R *such that for all* t *in* V *we have the inequality* $|u'(t)| \leq r|u(t)|$*. Then* $u(t) = 0$ *for all* t *in* I*.*

Proof. Define A to be the set $\{t \in I : u(t) = 0\}$. By our assumptions, A is nonempty, closed in I, and definable. By Lemma 2.4, it suffices to show that A is open.

Fix t_0 in A. By replacing $u(t)$ by $u(t - t_0)$, we may assume $t_0 = 0$. Let r and V be as in our hypotheses, and choose $\epsilon < 1/r$ such that $[-\epsilon, \epsilon] \subseteq V$. We show $[-\epsilon, \epsilon] \subseteq A$, which shows A is open. Set $s := \max\{|u(t)| : t \in [-\epsilon, \epsilon]\}$. Suppose there is a $t \in (0, \epsilon]$ such that $|u(t)| = s$. Then we can set

$$t_1 := \inf\{t \in (0, \epsilon] : |u(t)| = s\}.$$

In this case, $|u(t_1)| = s$ by continuity. Suppose for a contradiction that $t_1 \neq 0$.

Then, by the Mean Value Theorem, there is a $t_2 \in (0, t_1)$ such that $|u(t_1)| = |t_1 u'(t_2)|$. In this case we have

$$s = |u(t_1)| = |t_1 u'(t_2)| \leq r|t_1 u(t_2)| < |u(t_2)|,$$

a contradiction. The case where there is a $t \in [-\epsilon, 0)$ with $|u(t)| = s$ is done in the same way. □

We now prove Proposition 2.11:

Proof. Let $u : I \to R$ be the function defined by $u(t) := f(t) - g(t)$, and let t_0 be an arbitrary element of I. It suffices to find a V and an r as in the previous Lemma. Choose an $\epsilon > 0$ such that the closed interval $I_1 := [t_0 - \epsilon, t_0 + \epsilon]$ is contained in I. Set $a := \min(f(I_1) \cup g(I_1))$, $b := \max(f(I_1) \cup g(I_1))$, and $J_1 := [a, b]$. Consequently, if we take $V = (t_0 - \epsilon, t_0 + \epsilon)$ and

$$r = \max_{I_1 \times J_1} |D_2 F|\,,$$

then we are done: For $t \in V$,

$$|u'(t)| = |(f - g)'(t)| = |F(t, f(t)) - F(t, g(t))| \leq r|f(t) - g(t)| = r|u(t)|$$

by the Mean Value Theorem. □

The IVP allows us to prove versions of many other classical results from elementary differential topology. We conclude this Chapter with three important examples.

Theorem 2.13 (Inverse Function Theorem). *Let $f : U \to R^n$ be a definable function of class C^1 on an open subset U of R^n, and let a be a point in U where the matrix $(d_a f)$ is invertible. Then there exist definable open subsets U' and V' of R^n such that $a \in U' \subseteq U$ and $f|_{U'} : U' \to V'$ is a diffeomorphism of class C^1.*

Proof. This is exactly the proof of (2.11) given in Chapter 7 of [27]. □

Corollary 2.14 (Rank Theorem). *Suppose U and V are definable open subsets of R^n and R^m respectively and that $f : U \to V$ is a definable function of class C^1 with constant rank d. Then for any a in U there exist definable open neighborhoods U' of a and V' of $f(a)$, open subsets $\widetilde{U}$ of R^n and $\widetilde{V}$ of R^m, and definable diffeomorphisms $\varphi : U' \to \widetilde{U}$ and $\psi : V' \to \widetilde{V}$ of class C^1 such that*

$$\psi \circ f \circ \varphi^{-1}(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) = (x_1, \ldots, x_d, 0, \ldots, 0).$$

Proof. The proof of Theorem 7.8 in Lee [17] shows this after trivial modifications, including replacing the classical Inverse Function Theorem with the version above. □

Corollary 2.15 (Lagrange Multipliers). *Let U be a definable open subset of R^n, and let $f = (f_1, \ldots, f_m) : U \to R^m$ and $\varphi : U \to R$ be definable functions of class C^1. Suppose that there is a point $a \in U$ such that*

$$\varphi(a) \in \mathrm{bd}\{\varphi(b) : f(b) = f(a)\}.$$

Then the set of vectors $\{\nabla\varphi(a), \nabla f_1(a), \ldots, \nabla f_m(a)\}$ is linearly dependent.

Proof. Consider the map $\widetilde{\varphi} := f \times \varphi : U \to R^{m+1}$. If our desired conclusion fails then $\widetilde{\varphi}$ has constant rank $m + 1$ in some definable neighborhood U of a. Then by the Rank Theorem, there is an open neighborhood U' of a in R^n and an open subset V' of R^{m+1} such that $\widetilde{\varphi}$ maps U' onto V'. In other words, $(f(a), \varphi(a))$ is an interior point of V'. This is a contradiction. □

3 $\mathcal{R}$-manifold theory

Intuitively, a smooth $\mathcal{R}$-manifold M of dimension m should be a set which is locally (and definably) diffeomorphic to R^m. There are many ways to make this precise. We use a definition that achieves our ends quickly without loss of rigor.

Definition 3.1. A definable subset M of R^n is an **$\mathcal{R}$-manifold of dimension m and class C^k** if for every a in M there are open subsets U and V of R^n and a definable diffeomorphism $\varphi = (\varphi_1, \dots, \varphi_n) : U \to V$ of class C^k (called a **chart for M at a**) such that $a \in U$, $0 \in V$, $\varphi(a) = 0$, and

$$U \cap M = \{x \in R^n : \varphi_{m+1}(x) = 0, \dots, \varphi_n(x) = 0\}.$$

In the case when $m = n$, this simply means that M is a definable open subset of R^n. If T is a first-order theory with the IVP, then by a **T-manifold** we mean an $\mathcal{R}$-manifold where $\mathcal{R}$ is any model of T.

Note that in case $R = \mathbb{R}$, an $\mathcal{R}$-manifold is just a definable embedded submanifold of $\mathbb{R}^n$ with definable charts. (For standard definitions and equivalences, see [17].) From now on, all $\mathcal{R}$-manifolds are assumed to be of class at least C^1.

Extending the notation above, it is sometimes convenient to decompose a chart φ into the functions $\overline{\varphi} = (\varphi_1, \dots, \varphi_m)$ and $\widehat{\varphi} = (\varphi_{m+1}, \dots, \varphi_n)$. Then the set $U \cap M$ is equal to the zero set of the definable function $\widehat{\varphi}$. In addition, for each a in M we define the **tangent space** T_aM of M at a to be the vector subspace of R^n given by

$$T_aM := \ker d_a\widehat{\varphi}. \tag{3.1}$$

In the case $m = n$, we set $T_aM := R^m$. This notion is well-defined: Suppose ϕ and ψ are two charts for M at a on an open set U. Then $\widehat{\psi} \circ \left(\widehat{\phi}\right)^{-1}$ is a diffeomorphism between open subsets of R^{n-m}, and

$$d_a\widehat{\psi} = d_0\left(\widehat{\psi}\circ\left(\widehat{\phi}\right)^{-1}\right)d_a\widehat{\phi}$$

by the chain rule. Likewise, if $\sigma : W \to U$ is a map that is a diffeomorphism of class C^1 on an open subset W of R^n, and if we let $N = \sigma^{-1}(M)$, then

$$d_b\sigma(T_bN) = T_{\sigma(b)}M \tag{3.2}$$

for all b in N. We can conclude from this discussion that the dimension of an $\mathcal{R}$-manifold is uniquely determined and is equal to the dimension of its tangent space. Notice also that a definable subset N of an $\mathcal{R}$-manifold M is relatively open in M if and only if N is an $\mathcal{R}$-manifold of the same dimension as M.

Suppose now that M is of class C^k for some $k \geq 1$, and let $f : M \to R^l$ be a definable function. We say f is of **class** C^k on M if for each a in M there is a chart $\varphi : U \to V$ for M at a, such that the function

$$\overline{f} : \Pi_m(V \cap (R^m \times \{0\}^{n-m})) \to R^l$$

given by $\overline{f}(y) := f(\varphi^{-1}(y, 0, \ldots, 0))$ is of class C^k.

As in the introduction, we identify a 1-form $\omega = a_1dx_1 + \cdots + a_ndx_n$ on M with the vector field of its component functions $F_\omega = (a_1(x), ..., a_n(x))$. In this way, for each $a \in M$ and each $v \in T_aM$ we have $\omega(a)v = F(a) \cdot v$. We say ω is definable if and only the functions $a_i : M \to R$ are definable for $i = 1, \ldots, n$. Similarly, we say ω is of class C^k on M if the functions $a_i : M \to R$ are of class C^k for $i = 1, \ldots, n$. Finally, a 1-form ω is nonsingular on M if the vector field F is nowhere vanishing on M. Unless stated otherwise, all 1-forms under discussion are assumed to be nonsingular and of class C^1.

Before moving on, we prove a version of Lagrange Multipliers for $\mathcal{R}$-manifolds.

Proposition 3.2. *Suppose M is an $\mathcal{R}$-manifold of dimension m. Suppose M is contained in an open subset W of R^n and that $\mu : W \to R$ is a definable function* *of class C^1. Finally suppose that the function $\mu|_M$ has a local extremum at a point a. Then*

$$\nabla\mu(a) \in (T_aM)^\perp.$$

Proof. Let $\varphi : U \to V$ be a chart for M at a, so that M is locally the zero set of $\widehat{\varphi}$. Then by Corollary 2.15 we have

$$\nabla\mu(a) \in \langle \nabla\varphi_{m+1}(a), \ldots, \nabla\varphi_n(a)\rangle = (T_aM)^\perp$$

as desired. □

We next define the **rank** of f at a, denoted $\operatorname{rank}_a f$, to be the rank of the matrix $d_0\overline{f}$. Let us compute the rank of a linear transformation restricted to an $\mathcal{R}$-manifold.

Example 3.3. *Suppose $f : R^n \to R^m$ is linear. Let a be in M, and let $\varphi : U \to V$ be a chart for M at a. Then*

$$\operatorname{rank}_a(f|_M) = \operatorname{rank}_a(d_0\overline{f}) = \dim(d_0(f \circ \varphi^{-1})(R^m \times \{0\}^{n-m})) =$$

$$\dim(f(d_0\varphi^{-1}(R^m \times \{0\}^{n-m}))) = \dim(f(T_aM)).$$

Therefore, the rank of $f|_M$ at a is the dimension of the image of the tangent space T_aM.

Theorem 3.4 (Constant Rank Level Set Theorem). *Let M be an $\mathcal{R}$-manifold of dimension m in R^n. Let $f : M \to R^l$ be a definable function of class C^1 with constant rank equal to d. Then for each $b \in R^l$, the set $f^{-1}(b)$ is either empty or an $\mathcal{R}$-manifold of dimension $m - d$ (and class C^1).*

Moreover, suppose that f is defined on an open set containing M, that $f(a) = b$, and that $N = f^{-1}(b)$; then we have

$$T_aN = T_aM \cap \ker d_af.$$

Proof. By definition, there are open subsets U and V of R^n and a diffeomorphism $\varphi = (\varphi_1, \ldots, \varphi_n) : U \to V$ such that $a \in U$, $\varphi(a) = 0$, and $M \cap U = (\widehat{\varphi})^{-1}(0)$.

Since the function f has constant rank on $U \cap M$, the function

$$\widetilde{f} := f \circ \varphi^{-1}(y, 0, \ldots, 0)$$

has constant rank d on $\Pi_m(V \cap (R^m \times \{0\}^{n-m}))$. Thus by the Rank Theorem (Corollary 2.14), there are open subsets V' of R^m and W of R^l and diffeomorphisms $\overline{\sigma} : V' \to \widetilde{V}$ and $\psi : W \to \widetilde{W}$ such that $0 \in V'$ and

$$\psi \circ \widetilde{f} \circ \overline{\sigma}^{-1}(y) = (y_1, .., y_d, 0, \ldots, 0).$$

Let U' be the open set $\{x \in U : \overline{\varphi}(x) \in V'\}$, and consider the diffeomorphism $\sigma := (\overline{\sigma} \circ \overline{\varphi}, \widehat{\varphi})$ on U'. If we set $\rho := (\sigma_1, \ldots, \sigma_d)$ we see that $N \cap U'$ is precisely the set

$$\{x \in U' : \rho(x) = 0 \text{ and } \widehat{\varphi}(x) = 0\}.$$

Thus (after a permutation of coordinates) σ is a chart for N at a, and N is an $\mathcal{R}$-manifold of dimension $m - d$.

When f is defined on an open set containing M, we have the identity

$$\psi \circ f = (\rho(x), 0, \ldots 0)$$

on an open neighborhood of a. Thus $\ker d_a f = \ker d_a \rho$ since ψ is a diffeomorphism. Then by definition we have

$$T_a N = \ker d_a \rho \cap \ker d_a \widehat{\varphi} = \ker d_a f \cap T_a M,$$

finishing the proof. □

This motivates the following definition:

Definition 3.5. An $\mathcal{R}$-manifold M in R^n is said to be in **standard position** if for every strictly increasing map $\iota : \{1, \ldots, k\} \to \{1, \ldots, n\}$ there is a number $d = d(M, \iota)$ such that $\Pi_\iota|_M$ has constant rank d.

Corollary 3.6. *Let $n > 1$. Suppose a subset M of R^n is an $\mathcal{R}$-manifold of dimension m and class C^1 in standard position. Then for each $k < n$ and a in R^k the set $\{a\} \times M_a$ is either empty or an $\mathcal{R}$-manifold in standard position. In fact, for each increasing map $\iota : \{1, \dots, l\} \to \{k+1, \dots, n\}$, the rank of $\Pi_\iota|_{\{a\}\times M_a}$ does not depend on a.*

Proof. Fix a point (a, b) in M, and let d be the rank of $\Pi_k|_M$. Theorem 3.4 says that the set $\{a\} \times M_a$ is an $\mathcal{R}$-manifold of dimension $m - d$.

To see that $\{a\} \times M_a$ is in standard position, let ι be as in the statement of the Lemma. Let $\widehat{\iota}$ be the unique increasing function $\widehat{\iota} : \{1, \dots, k + l\} \to \{1, \dots, n\}$ such that

$$\widehat{\iota}(\{1, \dots, k + l\}) = \{1, \dots, k\} \cup \iota(\{1, \dots, l\}).$$

Then by Example 3.3, we get

$$\operatorname{rank}_{(a,b)} \Pi_\iota|_{\{a\}\times M_a} = \dim(\Pi_\iota(T_{(a,b)}(\{a\} \times M_a))).$$

Note that Theorem 3.4 implies that

$$T_{(a,b)}(\{a\} \times M_a) = \ker \Pi_k \cap T_{(a,b)}M.$$

So by looking at kernels, we see that

$$\ker(\Pi_\iota|_{T_{(a,b)}(\{a\}\times M_a)}) = \ker \Pi_\iota \cap T_{(a,b)}(\{a\} \times M_a)) =$$

$$\ker \Pi_\iota \cap \ker \Pi_k \cap T_{(a,b)}M = \ker(\Pi_{\widehat{\iota}}|_{T_{(a,b)}M}).$$

Suppose now that $\Pi_{\widehat{\iota}}|_M$ has constant rank e. Then since

$$\dim(\Pi_{\widehat{\iota}}(T_{(a,b)}M)) = \operatorname{rank}_{(a,b)}(\Pi_{\widehat{\iota}}|_M) = e,$$

we conclude that

$$\operatorname{rank}_{(a,b)} \Pi_\iota|_{\{a\}\times M_a} = (m - d) - (m - e) = e - d.$$

Since (a, b) was arbitrary, we are done. □

Corollary 3.7. *Let $n > 1$. Suppose a subset M of R^n is an $\mathcal{R}$-manifold of dimension m and class C^1 in standard position. Then for each $k < n$ and a in R^k the set M_a is either empty or an $\mathcal{R}$-manifold in standard position. In fact, for each increasing map $\iota : \{1, \ldots, l\} \to \{1, \ldots, n-k\}$, the rank of $\Pi_\iota|_{M_a}$ does not depend on a.*

Proof. From the previous Corollary, we have that $\{a\} \times M_a$ is an $\mathcal{R}$-manifold in standard position. It is then obvious that the (notationally more convenient) set $M_a \times \{a\}$ is an $\mathcal{R}$-manifold in standard position.

Let $\varphi = (\varphi_1, \ldots, \varphi_m) : U \to V$ be a chart for $M_a \times \{a\}$ at (b, a), and set $\bar{x} := (x_1, .., x_{n-k})$. Then the map $\Phi : U_a \to R^n$ given by

$$\Phi(\bar{x}) := (\varphi_1(\bar{x}, a), \ldots, \varphi_{n-k}(\bar{x}, a))$$

is a chart for M_a at b, showing that M_a is a manifold.

Let $\psi : U' \to V'$ be a chart for M_a at b. This time the map

$$\Psi(x) := (\psi(x_1, \ldots, x_{n-k}), x_{n-k+1} - a_1, \ldots, x_n - a_k)$$

is a chart for $M_a \times \{a\}$ at (b, a). Using this chart to calculate the tangent space, we see that

$$T_{(b,a)}(M_a \times \{a\}) = T_b M_a \times \{0\}^k.$$

For ι as above we have

$$\Pi_\iota(T_b M_a) = \Pi_\iota((T_b M_a) \times \{0\}^k) = \Pi_\iota(T_{(b,a)}(M_a \times \{a\})).$$

The result now follows from Example 3.3 and the previous Corollary. $\square$

We end this Chapter with a brief discussion of definable path connectedness.

Definition 3.8. A subset X of R^n is **definably path connected** if for all a and b in X there is an definable continuous function (path) $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. A **definably path connected component** of X is a maximal definably connected nonempty subset of X. We abbreviate this unwieldy term by calling it a **p-component**.

Remarks: (i) As expected, if X is definably path connected, then X is definably connected. (ii) If X is a subset of R^n and $x \in X$, then there is always a p-component of X containing x by Zorn's lemma. However, it is not clear that p-components of definable sets are definable. (Corollary 6.3 below deals with a special case of this.)

Proposition 3.9. *If M is an $\mathcal{R}$-manifold and c is in M, then there is an open set U containing c such that $U \cap M$ is definably path connected. Moreover, if M is of class C^k, then for any a and b in M, there is a definable path $\gamma : [0,1] \to M$ of class C^k such that $\gamma(0) = a$ and $\gamma(1) = b$.*

Proof. Let $\varphi : U \to V$ be a chart for M at a. By taking a small box around $\phi(a)$, we may assume that V is convex. Then for any a and b in U, the definable function given by $\gamma : [0,1] \to V$ by $\gamma(t) := t\varphi(b) + (1-t)\varphi(a)$ parametrizes the segment from $\varphi(a)$ to $\varphi(b)$. Consequently, the function $\varphi^{-1} \circ \gamma : [0,1] \to M$ is a definable path of class C^k connecting a and b. □

Corollary 3.10. *Let M be a definably connected $\mathcal{R}$-manifold. Then M is definably path connected if and only if some p-component of M is definable.*

Proof. By the previous Proposition, each p-component of M is open in M. Since the p-components of M are pairwise disjoint, each p-components is also closed in M. Hence if C is a definable p-component of M, then $C = M$ by Lemma 2.4. □

4 Background from o-minimality

One of the first important results in the model theory of ordered structures came from Tarski [26], who proved that the structure $\overline{\mathbb{R}}$ admits elimination of quantifiers. This means that all definable sets in this structure (commonly called semialgebraic sets) are given by boolean combinations of polynomial equations and inequalities over the reals. From this fact, it is easy to deduce that the semialgebraic subsets of $\mathbb{R}$ are simply finite unions of points and open (possibly unbounded) intervals. Another way to say this is that every $\overline{\mathbb{R}}$-definable subset of $\mathbb{R}$ is already definable in the reduct $\langle \mathbb{R}, < \rangle$.

In the 1980s, Van den Dries observed that this property on its own is strong enough to imply many of the finiteness results displayed by the collection of semialgebraic sets. Knight, Pillay, and Steinhorn in [23] and [15] subsequently extended this insight to arbitrary expansions of dense linear orders. Out of this study emerged the elegant theory of o-minimal structures. (See [27] for details.)

Definition 4.1. An expansion $\widetilde{\mathcal{R}}$ of a dense linear order $\langle R, < \rangle$ is called **o-minimal** if every definable subset of R is a finite union of points and open intervals with endpoints in $R \cup \{-\infty, \infty\}$.

It is immediate from the definition that the universe of an o-minimal structure is definably connected, and hence that every o-minimal structure has the IVP. From now on, we let $\widetilde{\mathcal{R}}$ be an o-minimal expansion of $\overline{\mathcal{R}}$. In this Chapter (and only in this Chapter), the term "definable" means definable in $\widetilde{\mathcal{R}}$.

The central feature of an o-minimal structure is that every definable set can be decomposed into cells. We define these terms in the next two definitions.

Definition 4.2. A definable subset C of R^n is a **cell** if

(i) $n = 1$ and C is a singleton or an open interval, or

(ii) $n > 1$ and there is a cell $B \subseteq R^{n-1}$, and definable continuous functions $f, g : B \to R$ with $f < g$ such that C is one of the following sets:

$$\Gamma(f), (f, g), (f, \infty) \text{ or } (-\infty, f).$$

(See the conventions following the introduction for the definitions of these sets.)

Definition 4.3. A partition $\mathcal{P}$ of R^n into finitely many cells is called a **decomposition** if one of the following holds

(i) $n = 1$ and $\mathcal{P}$ is a collection of the form

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty), \{a_1\}, \{a_2\}, \dots, \{a_k\}\}$$

(ii) $n > 1$ and the set $\{\Pi_{n-1}(A) : A \in \mathcal{P}\}$ is a decomposition of R^{n-1}.

We also say a partition $\mathcal{P}$ of R^n is **compatible** with a subset B of R^n if for every $A \in \mathcal{P}$ either $A \subseteq B$ or $A \cap B = \emptyset$.

The next Theorem is the fundamental theorem for o-minimal structures.

Theorem 4.4 (Cell Decomposition Theorem [27]). *Given any finite collection of definable subsets $A_1, \dots, A_l$ of R^n, there is a decomposition $\mathcal{P}$ of R^n that is compatible with A_i for each $i = 1, \dots, l$.*

Below we need a variant of this Theorem that demands the cells to be smooth in some sense. We recursively define a cell $C \subseteq R^n$ to be a C^k**-cell** if in Definition 4.2, the set B is a C^k-cell in R^{n-1} that is also an $\widetilde{\mathcal{R}}$-manifold of class C^k and the functions f and g are of class C^k on B. It follows from Proposition 4.6 below that every C^k-cell is an $\widetilde{\mathcal{R}}$-manifold of class C^k; thus, this assumption is ultimately superfluous. In this terminology, the cell decomposition theorem is still true if we fix a k and demand that each cell $A \in \mathcal{P}$ is a C^k-cell.

A straightforward induction on n yields the following Lemma:

Lemma 4.5. *Let N be a C^k-cell in R^n that is neither a singleton nor an open set. Then there is an $m \in \{1, ..., n-1\}$, an open C^k-cell N_1 in R^m, and a definable C^k-function $F : N_1 \to R^{n-m}$ such that N is the graph of F after a permutation of coordinates.*

The following Proposition shows that C^k-cells are C^k-diffeomorphic to sets of a very simple form. (See also Miller and Van den Dries [28].)

Proposition 4.6. *Let N be a C^k-cell in R^n. Then there is a natural number m, there are open sets U and V containing N and $R^m \times \{0\}^{n-m}$ respectively, and there is a definable diffeomorphism $\sigma : U \to V$ of class C^k such that*

$$\sigma(N) = R^m \times \{0\}^{n-m}.$$

Remark: For a C^1-cell N, the **tangent bundle of** N

$$TN := \{(x, v) \in N \times R^n : v \in T_x N\}$$

is definable by a formula that says "$(d_x\sigma)v = 0$."

Before giving the proof of Proposition 4.6, we derive another useful Corollary.

Corollary 4.7. *If a subset A of R^n is definable, then there is an $n' \geq n$ and a closed subset B of $R^{n'}$ such that $A = \Pi_n(B)$.*

Proof of Corollary 4.7. First off, when A is a cell of dimension m, then by Proposition 4.6 there is a definable homeomorphism $g : A \to R^m$. The result then follows if we let $n' := n + m$ and take B to be the graph $\Gamma(g)$ of g.

Now suppose the result holds for definable subsets A_1 and A_2 of R^n. That is, suppose there are n'_1 and n'_2 greater than n and closed definable subsets B_1 and B_2 of $R^{n'_1}$ and $R^{n'_2}$ respectively such that $A_1 = \Pi_n(B_1)$ and $A_2 = \Pi_n(B_2)$. Then the result holds for $A_1 \cup A_2$ by putting

$$n' := \max\{n'_1, n'_2\} \text{ and } B := B_1 \times R^{(n'-n'_1)} \cup B_2 \times R^{(n'-n'_2)}.$$

The Corollary now follows by induction and cell decomposition. □

To facilitate the proof of Proposition 4.6, we prove an intermediate Lemma.

Lemma 4.8. *Let N be an open C^k-cell in R^n. Then there is a definable diffeomorphism $\sigma : N \to R^n$ of class C^k.*

Proof. Define the diffeomorphism $\tau : (-1,1) \to R$ by

$$\tau(t) = \frac{t}{\sqrt{1-t^2}}.$$

Also, for any interval (a,b) in R define $\rho(a,b) : (a,b) \to (-1,1)$ by

$$\rho(a,b)(t) = \frac{2t-a-b}{b-a}.$$

So, for example, the interval $(-\infty, b)$ is diffeomorphic to R via the function $\tau \circ \rho(-1,0) \circ \tau^{-1}(t-b)$.

In higher dimensions we use induction; here is one case: Suppose $N = (f,g)$, where $f, g : N_1 \to R$ are C^k-functions and N_1 is an open C^k-cell in R^{n-1}. By induction, there is a diffeomorphism $\sigma_1 : N_1 \to R^{n-1}$ of class C^k. Let $\overline{x} = (x_1, \ldots, x_{n-1})$ and define $\sigma : N \to R^n$ by

$$\sigma(\overline{x}, x_n) = (\sigma_1(\overline{x}), \rho(f(\overline{x}), g(\overline{x}))(x_n)).$$

The other cases are similar. □

Now the proof of Proposition 4.6:

Proof. If N is a singleton, then the result is trivial. If N is open, then the result follows from Lemma 4.8. Thus we may invoke Lemma 4.5 and assume that N is the graph of a C^k-function

$$F = (F_{m+1}, \ldots, F_n) : N_1 \to R^{n-m},$$

where N_1 is an open C^k-cell in R^m. By Lemma 4.8, there is a definable diffeomorphism $\sigma_1 : N_1 \to R^m$ of class C^k. We may then take $U = N_1 \times R^{n-m}$ and

$$\sigma(x) = (\sigma_1(\overline{x}), x_{m+1} - F_{m+1}(\overline{x}), \ldots, x_n - F_n(\overline{x})),$$

where $\overline{x} = (x_1, \ldots, x_m)$. □

All o-minimal structures admit a well-behaved **dimension theory**. (See Chapter four of [27].) In an o-minimal expansion of a field, the dimension $\dim(C)$ of a cell C given by o-minimality agrees with the dimension of C as an $\widetilde{\mathcal{R}}$-manifold; so for our purposes, we take this to be the definition of $\dim(C)$. We then define the dimension of an arbitrary definable set A by setting

$$\dim(A) := \max\{\dim(C) : C \text{ is a cell contained in } A\}.$$

If A is empty, we put $\dim(A) := -\infty$.

This dimension satisfies many attractive properties. For example, if A and B are two definable subsets of R^n, then $\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$. Another property we use below is the following.

Proposition 4.9. *Let A be a definable subset of R^{m+n}. For d in $\{0, 1, \ldots, n\}$, let $A(d) := \{a \in R^m : \dim(A_a) = d\}$. Then $A(d)$ is definable and*

$$\dim\left(\bigcup_{a \in A(d)} \{a\} \times A_a\right) = \dim(A(d)) + d.$$

For the rest of this Chapter, we prove some technical Lemmas that are used in the proof of Theorem 1.7. Let U be an open subset of R^n, and fix a finite tuple $\Omega := (\omega_1, \ldots, \omega_q)$ of definable 1-forms ω_i on U. For $i = 1, \ldots, q$ we also write F_i for the vector field F_{ω_i} associated with ω_i.

Definition 4.10. Let N be an $\mathcal{R}$-manifold contained in U. The tuple Ω is **transverse to** N if

$$\dim\left(T_a N \cap \bigcap_{i=1,\ldots,q} \ker(\omega_i(a))\right) = \dim(N) - q$$

for all $a \in N$. If $N = R^n$, we simply say Ω is **transverse**. For a subset J of $\{1, \ldots, q\}$, we write $\Omega_J := (\omega_j)_{j \in J}$. The tuple Ω_J is called a **basis of Ω along** N if Ω_J is transverse to N and

$$T_a N \cap \bigcap_{i=1,\ldots,q} \ker(\omega_i(a)) = T_a N \cap \bigcap_{i \in J} \ker(\omega_i(a))$$

for all $a \in N$.

The next Lemma is an amalgam of Lemmas 2.1 and 2.8 from [25].

Lemma 4.11. *Let A be a definable subset of R^n. Then for any natural number k, there is a decomposition $\mathcal{P}$ of R^n into C^k-cells such that $\mathcal{P}$ is compatible with both A and U and satisfies the following property: Whenever a cell N in $\mathcal{P}$ is a subset of A and whenever J is a subset of $\{1, \dots, q\}$, there exists a subset J' of J such that $\Omega_{J'}$ is a basis of Ω_J along C.*

Proof. By induction on $d := \dim(A)$. The case $d = 0$ is trivial. Assume $d > 0$ and that the Lemma holds for lower values of d. Using C^k-cell decomposition and the inductive hypothesis, we reduce to the case that A is a C^k-cell of dimension d.

Now for an element a of A and a subset J of $\{1, \dots, q\}$, we write

$$T_a(\Omega_J) = T_a A \cap \bigcap_{i \in J} \ker(\omega_i(a)).$$

For each subset J of $\{1, \dots, q\}$ and each subset J' of J, we also define the set

$$A(J, J') := \{a \in A : T_a(\Omega_J) = T_a(\Omega_{J'}) \text{ and } \dim(T_a(\Omega_{J'})) = d - |J'|\}.$$

Since the tangent bundle TA is definable and each ω_i is definable, the set $A(J, J')$ is definable too. Let $\mathcal{P}_1$ be a C^k-cell decomposition compatible with each set $A(J, J')$ and the set U. Put $B := \bigcup\{C \in \mathcal{P}_1 : \dim(C) < d\}$, and note that B is definable and that $\dim(B) < d$. Then by the inductive hypothesis, the Lemma holds with B in place of A, producing another C^k-cell decomposition $\mathcal{P}_2$. Let $\mathcal{P}$ be the cell decomposition given by

$$\mathcal{P} := \{C \in \mathcal{P}_1 : \dim(C) = d\} \cup \{C \in \mathcal{P}_2 : C \subseteq B\}.$$

We claim that this $\mathcal{P}$ works: Fix a subset J of $\{1, \dots, q\}$, and let C be in $\mathcal{P}$ with $C \subseteq A$. If $\dim(C) = d$ then C is a relatively open subset of A, and hence $T_a C = T_a A$ for all a in C. Since there is a J' such that $C \subseteq A(J, J')$, the tuple

$\Omega_{J'}$ is a basis of Ω_J along C. On the other hand, if $\dim(C) < d$, then C is in $\mathcal{P}_2$ and the result follows. □

As our last task of this chapter, we prove a lemma with conceptual origins in Morse theory. We use it to lower the dimension in the inductive proof of our Khovanskii Theory (Theorem 1.7).

Definition 4.12. Let $M = R^m \times \{0\}^{n-m}$. A **positive $\mathcal{R}$-form** for M is an definable continuous function $\mu : R^n \to [0, \infty)$ such that for each positive r, the set $M \cap \mu^{-1}([0, r])$ is bounded in R^n.

Example 4.13. *For positive elements $u_1, \ldots u_m$ of R, the function* $\mu_u : R^n \to R$ *given by*

$$\mu_u(x) := \sum_{i=1}^{m} u_i x_i^2$$

is a positive $\widetilde{\mathcal{R}}$-form for M.

We use this observation in the next Lemma.

Lemma 4.14. *Suppose that M is contained in U and that Ω is transverse to M. Suppose also that $q < m$. Then there is a positive $\widetilde{\mathcal{R}}$-form μ for M of class C^1 such that the definable set*

$$B := \{a \in M : \nabla \mu(a) \in \langle F_1(a), \ldots, F_q(a), e_{m+1}, \ldots, e_n \rangle\}$$

has dimension strictly less than m.

Proof. For $u \in (0, \infty)^m$, let μ_u be as above. Notice also that

$$\nabla \mu_u(a) \in \langle F_1(a), \ldots, F_q(a), e_{m+1}, \ldots, e_n \rangle$$

if and only if

$$\nabla \mu_u(a) \in \langle F_1(a), \ldots, F_q(a) \rangle.$$

So put

$$\mathcal{D}_u := \{a \in M : \nabla \mu_u(a) \in \langle F_1(a), \ldots, F_q(a) \rangle\}.$$

Assume for a contradiction that $\dim(\mathcal{D}_u) = m$ for all $u \in (0,\infty)^m$. Then it follows from Proposition 4.9 that $\dim(\mathcal{D}) = 2m$ where $\mathcal{D}$ is the definable set

$$\mathcal{D} := \{(u,a) \in (0,\infty)^m \times M : a \in \mathcal{D}_u\}.$$

Thus there are nonempty open subsets V of $(0,\infty)^m$ and W of R^m such that $V \times W \times \{0\}^{n-m} \subseteq \mathcal{D}$. Fix a in $W \times \{0\}^{n-m}$ such that $a_i \neq 0$ and $a_i \neq 1$ for $i = 1,\dots,m$, and consider the definable function $G : M \times R^m$ of class C^1 given by

$$G(x_1,\dots,x_m) = \left(x_1^2,\dots,x_m^2\right)$$

We see that $\nabla\mu_u(a) = (d_aG)u$. Since the linear map d_aG has rank m, the set $\langle \nabla\mu_u(a) : u \in V\rangle$ is an R-linear subspace of $\langle F_1(a),\dots,F_q(a)\rangle$ of dimension m. This contradicts that $q < m$. □

5 Khovanskii theory with the IVP

Recall from the introduction that $\overline{\mathbb{R}}$ denotes the field of real numbers and that $\widetilde{\mathbb{R}}$ is an o-minimal expansion of $\overline{\mathbb{R}}$. An essential ingredient in the proof of Theorem 1.6 is the following version of Khovanskii Theory due to Van den Dries. (See [25].)

Theorem 5.1. *Let $\omega_1, \ldots, \omega_q : U \to R^n$ be nonsingular 1-forms of class C^1 on a common open subset U of $\mathbb{R}^n$ that are definable in $\widetilde{\mathbb{R}}$. Let A be a subset of $\mathbb{R}^n$ that is also definable in $\widetilde{\mathbb{R}}$. Then there is a natural number K such that whenever L_i is a Rolle leaf of $\omega_i = 0$ for each $i = 1, \ldots, q$, then $A \cap L_1 \cap \cdots \cap L_q$ is a union of fewer than K connected manifolds.*

As mentioned earlier, we can generalize this to Theorem 1.7 using tools that derive from the IVP alone. For the most part, the proof from [25] goes through after the definitions have been appropriately adapted. However, at a crucial point of that proof the fact that a connected manifold is path connected is used. Since we do not have an analogous fact for $\mathcal{R}$-manifolds, we are forced to strengthen the Rolle property. On the other hand, we do have such a fact for T^{proj}-manifolds (by Corollary 6.3), an observation that is used later.

Here are our standing assumptions: We have a real closed ordered field $\overline{\mathcal{R}} := \langle R, +, \cdot, <, 0, 1\rangle$ and an o-minimal expansion $\widetilde{\mathcal{R}}$ of $\overline{\mathcal{R}}$. We take an expansion $\mathcal{R}$ of $\widetilde{\mathcal{R}}$ with the IVP. In contrast with the previous Chapter, "definable" now means definable in $\mathcal{R}$. In case a set is definable in $\widetilde{\mathcal{R}}$, we say it is $\widetilde{\mathcal{R}}$-definable to emphasize this point. Also, all $\mathcal{R}$-manifolds and diffeomorphisms are assumed to be at least of class C^1. Finally, all 1-forms are assumed to be nonsingular and of class C^1 on their domains.

The following two definitions take the place of Definition 1.5 in our context.

Definition 5.2. Let ω be a 1-form on an open subset U of R^n. We say that an $\mathcal{R}$-manifold of dimension $n-1$ and class C^1 is an **integral $\mathcal{R}$-manifold of $\omega = 0$** if M is a subset of U and $T_aM = \ker(\omega(a))$ for all a in M. An **$\mathcal{R}$-leaf of** $\omega = 0$ is a definably connected integral $\mathcal{R}$-manifold of $\omega = 0$ that is relatively closed in U.

Definition 5.3. Let ω be a 1-form on an open subset U of R^n. An $\mathcal{R}$-leaf L of $\omega = 0$ is called **Rolle** if L satisfies the following additional property:

> [$\mathcal{R}$-Rolle Property] For every definably connected $\mathcal{R}$-manifold C of dimension 1 and class C^1 that is contained in U, either $|C \cap L| \leq 1$ or there is an a in C such that $T_aC \subseteq \ker(\omega(a))$.

Remark: Under the condition that $\mathcal{R}$ is a model of T^{proj}, Corollary 6.3 below tells us that L is a Rolle $\mathcal{R}$-leaf of $\omega = 0$ if and only if this more familiar Rolle property holds:

> [Alternate $\mathcal{R}$-Rolle Property] For each definable C^1 curve $\gamma : [0,1] \to U$ with $\gamma(0) \in L$ and $\gamma(1) \in L$, there is a $t \in [0,1]$ such that $\omega(\gamma(t))\gamma'(t) = 0$.

The rest of this chapter is devoted to the proof of Theorem 1.7. First some Lemmas.

Lemma 5.4. *Let U be an $\widetilde{\mathcal{R}}$-definable open subset of R^n, and let ω be an $\widetilde{\mathcal{R}}$-definable 1-form on U. Let L be an integral $\mathcal{R}$-manifold of $\omega = 0$ that is closed in U. Suppose a subset C of U is a definably connected $\mathcal{R}$-manifold of dimension at most $n-1$ and that $T_xC \subseteq \ker(\omega(x))$ for all x in C. Then either $C \cap L = \emptyset$ or $C \subseteq L$.*

Proof. Write $\omega = a_1dx_1 + \cdots + a_ndx_n$ with each $a_i : U \to R$ of class C^1. Assume that $C \cap L$ is nonempty. Since $C \cap L$ is definable and closed in C, it suffices to show $C \cap L$ is open in C by Lemma 2.4.

Fix b' in $C \cap L$. Permuting coordinates if necessary, we may assume that $a_n(b) \neq 0$ for all b in some definable open neighborhood V of b'. Then for all b in V, we have $\Pi_{n-1} T_b(V \cap L) = R^{n-1}$; so Π_{n-1} has constant rank $n-1$ on $V \cap L$ by Example 3.3. It then follows from Theorem 3.4 that, after further shrinking V if necessary, the set $V \cap L$ is the graph of a function $f : W \to R$ of class C^1, where $W := \Pi_{n-1}(V)$. Moreover, the function f satisfies the following equations:

$$D_i f(x) = -\frac{a_i}{a_n}(x, f(x)), \text{ for } x \in W \text{ and } i = 1, \dots, n-1. \tag{5.1}$$

By Lemma 3.9 there is a definable and definably path connected relatively open subset C' of $C \cap V$ containing b'. It suffices to show C' is contained in the graph $\Gamma(f)$ of f. Let b'' be a point of C' distinct from b'. We show b'' is in $\Gamma(f)$. Let $s > 0$ and let $\gamma = (\gamma_1, \dots, \gamma_n) : (-s, 1+s) \to C'$ be a path of class C^1 such that $\gamma(0) = b'$ and $\gamma(1) = b''$. Then we have $\omega(\gamma(t))\gamma'(t) = 0$ for all t in the interval $(-s, 1+s)$. This means that

$$\gamma_n'(t) = -\frac{a_1}{a_n}(\delta(t), \gamma_n(t)) \cdot \gamma_1'(t) - \cdots - \frac{a_{n-1}}{a_n}(\delta(t), \gamma_n(t)) \cdot \gamma_{n-1}'(t)$$

for all t in $(-s, 1+s)$, where $\delta := (\gamma_1, \dots, \gamma_{n-1}) : (-s, 1+s) \to W$. If we also put $h(t) := f(\delta(t))$, then by (5.1) the function h satisfies the equation

$$h'(t) = -\frac{a_1}{a_n}(\delta(t), h(t)) \cdot \gamma_1'(t) - \cdots - \frac{a_{n-1}}{a_n}(\delta(t), h(t)) \cdot \gamma_{n-1}'(t)$$

for all t in $(-s, 1+s)$. Since $h(0) = b_n = \gamma_n(0)$, it follows from Proposition 2.11 that $\gamma_n = h$. Consequently, $b'' = \gamma(1) = (\delta(1), h(1)) \in \Gamma(f)$ which finishes the proof. □

We now discuss the pullback of a 1-form.

Definition 5.5. Let U and V be open subsets of R^n, and let $\sigma : V \to U$ be a definable diffeomorphism. Suppose that ω is a definable 1-form on U. Then the **pullback** $\sigma^*\omega$ is the 1-form on V given by

$$\sigma^*\omega(a)v := \omega(\sigma(a)) d_a \sigma v \tag{5.2}$$

for all $a \in V$ and $v \in T_a V$.

Lemma 5.6. *Let U and V be definable open subsets of R^n, and let $\sigma : V \to U$ be a definable diffeomorphism. Let M be an $\mathcal{R}$-manifold contained in U, and let ω be a definable 1-form on U. If M is an integral $\mathcal{R}$-manifold of $\omega = 0$, then $\sigma^{-1}(M)$ is an integral $\mathcal{R}$-manifold of the 1-form $\sigma^*\omega$.*

Proof. Choose b in $\sigma^{-1}(M)$. Since they are vector spaces of the same dimension, it suffices to show that $T_b(\sigma^{-1}(M)) \subseteq \ker(\sigma^*\omega(b))$. Choose v in $T_b\sigma^{-1}(M)$. By (3.2), we have that $(d_b\sigma)v \in T_{\sigma(b)}M$. Thus,

$$\sigma^*\omega(b)v = \omega(\sigma(b))d_b\sigma v = 0.$$

That is, $v \in \ker(\sigma^*\omega(b))$. □

Lemma 5.7. *Let U and V be definable open subsets of R^n, and let $\sigma : V \to U$ be a definable diffeomorphism. Let ω be a definable 1-form on U, and let L be a Rolle $\mathcal{R}$-leaf of $\omega = 0$. Then $\sigma^{-1}(L)$ is a Rolle $\mathcal{R}$-leaf of $\sigma^*\omega = 0$.*

Proof. By the previous Lemma, it suffices to verify the $\mathcal{R}$-Rolle property. Suppose C is a definably connected $\mathcal{R}$-manifold of dimension 1 contained in V that contains two distinct points of $\sigma^{-1}(L)$. Then $\sigma(C)$ is definably connected, contained in U, and meets two distinct points of L. Hence there is an a in $\sigma(C)$ such that $T_a\sigma(C) \subseteq \ker(\omega(a))$. Let $b = \sigma^{-1}(a)$, and let $v \in T_bC$. Then by (3.2), we have $d_b\sigma v \in T_a\sigma(C)$. Hence,

$$\sigma^*\omega(b)v = \omega(\sigma(b))d_b\sigma v = 0.$$

In other words, we have $T_bC \subseteq \ker(\sigma^*\omega(b))$ as required. □

Lemma 5.8. *Let σ be as in the previous Lemma, and suppose N is an $\mathcal{R}$-manifold contained in U. Suppose moreover that $\Omega := (\omega_1, \ldots, \omega_q)$ is a finite tuple of 1-forms on U that is transverse to N. (See Definition 4.10.) Then the tuple of pullbacks $\sigma^*\Omega := (\sigma^*\omega_1, \ldots, \sigma^*\omega_q)$ is transverse to $\sigma^{-1}(N)$.*

Proof. This follows from the fact that $d_a\sigma$ is a linear isomorphism for all a in M. □

For the remainder of this Chapter, fix a tuple $\Omega = (\omega_1, \ldots, \omega_q)$ of $\widetilde{\mathcal{R}}$-definable 1-forms on a common open subset U of R^n. Again we let F_i be shorthand for F_{ω_i}, the vector field associated with ω_i for $i = 1, \ldots, q$.

Lemma 5.9. *Let N be an $\mathcal{R}$-manifold contained in U, and let Ω_J be a basis of Ω along N. For each $i = 1, \ldots, q$, let L_i be a definably connected integral $\mathcal{R}$-manifold of $\omega_i = 0$, and write*

$$W := N \cap \bigcap_{i \in \{1,\ldots,q\}} L_i \quad \text{and} \quad W_J := N \cap \bigcap_{i \in J} L_i.$$

Then W_J is either empty or an $\mathcal{R}$-manifold of dimension $\dim(N) - |J|$. Also, for each a in M, the tangent space $T_a W_J$ is given by

$$T_a W_J := T_a N \cap \bigcap_{i \in J} \ker(\omega_i(a)).$$

Moreover, if W_J has only finitely many components, then each component of W is a component of W_J and an $\mathcal{R}$-manifold of dimension $\dim(N) - |J|$.

Proof. Assume W_J is not empty. Our first objective is to show that W_J is an $\mathcal{R}$-manifold of dimension $\dim(N) - |J|$. We may assume that $|J| = 1$ and write L for the integral $\mathcal{R}$-manifold of $\omega = 0$. Let a be in $N \cap L$, and let $\varphi : U \to V$ be a chart for L at a. Let $M = \varphi(U \cap N)$ and set $m := \dim(M)$. Notice also that the kernel of the pullback $(\phi^{-1})^* \omega(a)$ is equal to the orthogonal complement of the vector e_n for all a in V. Thus it suffices to show that $M' := M \cap (R^{n-1} \times \{0\})$ is an $\mathcal{R}$-manifold of dimension $m - 1$ and that $T_0 M' = T_0 M \cap e_n^\perp$.

Let Π be the projection $x \mapsto x_n$. By transversality, we have

$$m - 1 = \dim(T_0 M \cap e_n^\perp) = \dim(T_0 M \cap \ker \Pi) = m - \operatorname{rank}(\Pi|_M).$$

In other words, the function $\Pi|_M$ has rank 1 at the origin. After shrinking V if necessary, we may assume that $\Pi|_M$ has constant rank 1. Thus by Theorem 3.4, the set $M' = (\Pi|_M)^{-1}(0)$ is an $\mathcal{R}$-manifold of dimension $m-1$ with tangent space $T_0 M \cap \ker \Pi$. This completes our first objective.

For the second assertion, we show that every component of W_J that meets W is a component of W. Assume that W_J has only finitely many components and let C be one of them. Thus, by Proposition 2.6, C is definable and relatively open in W_J. As a result, the set C is itself an $\mathcal{R}$-manifold. Now choose some j in $\{1, \ldots, q\} \backslash J$ such that the intersection $C \cap L_j$ is nonempty. Since Ω_J is a basis of Ω along N, for all a in C we have

$$T_a C \subseteq \ker(\omega_j(a)).$$

Now we apply Lemma 5.4 to conclude $C \subseteq L_j$. Since j was arbitrary, C is a subset—and hence a component—of W. □

Proposition 5.10. *Let N be an $\widetilde{\mathcal{R}}$-definable C^1-cell of dimension m contained in U, and suppose that $q < m$ and that Ω is transverse to N. Then there is an $\widetilde{\mathcal{R}}$-definable closed subset B of N with $\dim(B) < m$ such that whenever L_i is a Rolle $\mathcal{R}$-leaf of $\omega_i = 0$ for each i, we have one of the following cases:*

Case 1: Either $N \cap L_1 \cap \cdots \cap L_q$ has finitely many components, and each component of $N \cap L_1 \cap \cdots \cap L_q$ meets B,

Case 2: or $N \cap L_1 \cap \cdots \cap L_q$ has infinitely many components, and infinitely many of those components meet B.

Proof. First note that by Lemma 5.9, the set $N \cap L_1 \cap \cdots \cap L_q$ is always either empty or an $\mathcal{R}$-manifold of dimension $m - q$. By Corollary 4.6, there are open subsets $\widetilde{U}$ and $\widetilde{V}$ of R^n containing $R^m \times \{0\}^{n-m}$ and N respectively, and a diffeomorphism $\sigma : \widetilde{U} \to \widetilde{V}$ of class C^1 such that $\sigma(R^m \times \{0\}^{n-m}) = N$. Thus replacing the cell N by $R^m \times \{0\}^{n-m}$, the open set U by $\widetilde{U} \cap \sigma^{-1}(U)$, and each ω_i by the pullback $\sigma^*(\omega_i|_{\widetilde{V}})$, we may assume that $N = R^m \times \{0\}^{n-m}$. Let B and μ be the set and positive $\mathcal{R}$-form for N given by Lemma 4.14. We claim that this B works:

Suppose first that $N \cap L_1 \cap \cdots \cap L_q$ has finitely many components. Then each component C is definable and closed in N by Proposition 2.6. If we choose $r > 0$, then $C \cap \mu^{-1}[0, r]$ is closed and bounded in R^n. Thus $\mu|_C$ assumes a minimum value, say at the point a. By Proposition 3.2 and Lemma 5.9, this implies that

$$\nabla \mu_u(a) \in T_a(N \cap L_1 \cap \cdots \cap L_q)^\perp = \langle F_1(a), \ldots, F_q(a), e_{m+1}, \ldots, e_n \rangle,$$

finishing this case.

Now suppose $N \cap L_1 \cap \cdots \cap L_q$ has infinitely many definably connected components. We recursively produce infinitely many distinct components $C_1, C_2, \ldots$ each of which meets B, and definable sets $V_0 \supsetneq V_1 \supsetneq V_2 \ldots$ satisfying the following properties:

(1) Each V_i is an open and closed subset of $N \cap L_1 \cap \cdots \cap L_q$.

(2) Each V_i contains infinitely many definably connected components of $N \cap L_1 \cap \cdots \cap L_q$.

(3) And $C_i \cap V_j = \emptyset$ whenever $j > i$.

To start, put $V_0 := N \cap L_1 \cap \cdots \cap L_q$. Given V_i, we show how to produce V_{i+1}. Since V_i is not definably connected, there are definable open disjoint U_1 and U_2 such that $V_i \subseteq U_1 \cup U_2$ and both $V_1 \cap U_1$ and $V_1 \cap U_2$ are nonempty. Now one of U_1 or U_2 contains infinitely many components of $N \cap L_1 \cap \cdots \cap L_q$. We may assume this to be U_2, and we then set $V_{i+1} := V_i \cap U_2$. Now $V_i \cap U_1$ is a closed subset of $N \cap L_1 \cap \cdots \cap L_q$. Thus $\mu|_{V_i \cap U_1}$ assumes a minimum value. Arguing as in the previous case, the set B meets $V_i \cap U_1$. Thus B meets some component C of $V_i \cap U_1$. Take $C_{i+1} := C$ and we are done. □

Finally the proof of Theorem 1.7:

Proof. We proceed by induction on $d := \dim(A)$ and q. The cases $d = 0$ and $q = 0$ being trivial, we assume $d > 0$ and $q > 0$ and that the result holds for lower values of d or q. By Lemmas 4.11 and 5.9, it suffices to consider the case that A is an $\widetilde{\mathcal{R}}$-definable C^1-cell contained in U and Ω is transverse to A. Note that then $d \geq q$. For $i = 1, .., q$, we let L_i be a definable Rolle $\mathcal{R}$-Leaf of $\omega_i = 0$, and we put $L := L_1 \cap ... \cap L_q$. We now proceed by cases.

Case $d > q$: Let B be a closed definable subset of A with the property described in Proposition 5.10. In particular, the dimension of B is less than d. Then by the inductive hypothesis there is a natural number K, independent of the particular Rolle $\mathcal{R}$-leaves chosen, such that $B \cap L$ has fewer than K components. Now Proposition 5.10 tells us that the set $A \cap L$ has only finitely many components and each component meets B. It follows that $A \cap L$ has fewer than K components. The fact that each component of $A \cap L$ is an $\mathcal{R}$-manifold is a consequence of Lemma 5.9.

Case $d = q$: Put $L' := L_1 \cap \cdots \cap L_{q-1}$, and notice that $\dim(A \cap L') = 1$. By the inductive hypothesis, there is a natural number K such that the $\mathcal{R}$-manifold $A \cap L'$ has fewer than K components. Let C be a component of $A \cap L'$, and observe that C is an $\mathcal{R}$-manifold. Now if $|C \cap L_q| > 1$, then there is an a in C such that $T_a C \subseteq \ker(\omega_q(a))$ by the $\mathcal{R}$-Rolle property. But this contradicts that ω_q is transverse to $A \cap L'$. We must conclude that $|C \cap L_q| \leq 1$ for each componenet C of $A \cap L'$, and consequently that $|A \cap L' \cap L_q| < K$. $\square$

6 Tools from the theory of $\langle \overline{\mathbb{R}}, \mathbb{Z} \rangle$

Recall from the introduction that $\mathbb{R}_{\text{proj}} := \langle \overline{\mathbb{R}}, \mathbb{Z} \rangle$ and that T^{proj} is the theory of $\mathbb{R}_{\text{proj}}$. We shall also denote the language $\{+, \cdot, 0, 1, <, \mathbb{Z}\}$ by $\mathscr{L}_{\text{proj}}$.

It is well known that the definable sets in $\mathbb{R}_{\text{proj}}$ are precisely the projective sets of descriptive set theory. (This is exercise 37.6 in Kechris [13] for example.) Moreover, Gödel's famous Incompleteness Theorem [8] implies that T^{proj} has no recursive axiomatization. For these and other reasons, the structure $\mathbb{R}_{\text{proj}}$ is usually considered to be too "wild" for most model-theoretic purposes.

Below we find a use for such wild structures. Assume that $\mathcal{R}$ is a model of T^{proj}. We shall utilize $\mathcal{R}$ as a workspace out of which to carve o-minimal reducts. Indeed, though the definable sets in $\mathcal{R}$ can be very complicated, the structure $\mathcal{R}$ still has the IVP and hence satisfies the results of Chapter 2.

In this Chapter, we develop the tools that are required to make the main arguments from [25] work in $\mathcal{R}$. In particular, we state here an analogue of the classical Baire category theorem, and we define the Hausdorff limit of a definable sequence of closed and bounded subsets of R^n. This Chapter may also be regarded as a catalog of the results from $\mathbb{R}_{\text{proj}}$ that are at present required to prove Theorem 1.8. We hope that this list will be shortened in future research.

These results are all obtained for $\mathcal{R}$ in the same way: We exploit the fact that a sentence of first-order logic is true in $\mathcal{R}$ if and only if it is true in $\mathbb{R}_{\text{proj}}$. We shall argue that the results we need can be expressed as $\mathscr{L}_{\text{proj}}$-sentences that are true classically in $\mathbb{R}_{\text{proj}}$. Due to a resemblance with the transfer principle of nonstandard analysis, we refer to this method as a **transfer** argument. (See Henson [9].) In short, the results in this Chapter are "transferred from $\mathbb{R}_{\text{proj}}$ to $\mathcal{R}$." Arguments of this kind are routine in model theory, and many of the items below have likely been observed previously.

We begin by surveying the expressive power of $\mathbb{R}_{\text{proj}}$. The fundamental insight here comes from Gödel, who in Proposition VII of [8] shows that every recursive function $f : \mathbb{N} \to \mathbb{N}$ is definable. As a special case, for each natural number p the function $m \mapsto p^m$ is definable. From this one fact, we derive the following far-reaching consequences:

Proposition 6.1. *Let $n \geq 1$.*

(A) There is a definable (without parameters) subset X^n of $\mathbb{R}^{n+1}$ such that $\{X^n_r : r \in \mathbb{R}\}$ is equal to the collection of all subsets of $\mathbb{N}^n$.

(B) There is a definable (without parameters) subset Y^n of $\mathbb{R}^{n+1}$ such that $\{Y^n_r : r \in \mathbb{R}\}$ is equal to the collection of all open subsets of $\mathbb{R}^n$. Consequently, the same is true with "open" replaced by "closed."

(C) Let $m \geq 1$. There is a definable (without parameters) subset $Z^{m,n}$ of $\mathbb{R}^{m+n+1}$ such that $\{Z^{m,n}_r : r \in \mathbb{R}\}$ is equal to the collection of all graphs of continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. Moreover, the same is true for all functions from $\mathbb{N}^m$ to $\mathbb{R}^n$.

Remark: Using Matijasevich's theorem on Diophantine sets, one can definably obtain all open subsets of $\mathbb{R}^n$ from a single polynomial equation over $\mathbb{Z}$. For a precise statement of this result see (2.6) of [27]. The following elementary proof was provided by Alf Dolich.

Proof. (A) We first do the case $n = 1$. Let X^1 be the set given by

$$X^1 := \{(m, r) \in \mathbb{N} \times \mathbb{R} : \lfloor 10^m r \rfloor \equiv 1 \mod 10\}$$

where $\lfloor r \rfloor$ is the greatest integer less than or equal to r. Then for each subset S of $\mathbb{N}$, the set S equals X^1_r where $r = \sum_{m \in S} 10^{-m}$.

For the case $n > 1$, let $f : \mathbb{N}^n \to \mathbb{N}$ be a definable injection. (A standard one is the function $(m_1, \ldots, m_n) \mapsto 2^{m_1} 3^{m_2} \cdots {p_n}^{m_n}$ where p_n is the n-th prime.) Taking X^1 as in the previous case, the set

$$X^n := \{(m_1, \ldots, m_n, r) \in \mathbb{N}^n \times \mathbb{R} : f(m_1, \ldots, m_n) \in X^1_r\}$$

satisfies our requirements.

(B) The case of closed sets follows immediately from the case for open sets, which we prove now. For $\overline{q} = (q_0, \ldots, q_{n-1})$ in $\mathbb{Q}^n$ and m in $\mathbb{N}$, put

$$B_{\overline{q},m} := \{(s_0, \ldots, s_{n-1}) \in \mathbb{R}^n : m|q_i - s_i| < 1 \text{ for } i = 0, \ldots, n-1\}.$$

We use the fact that $\{B_{\overline{q},m} : \overline{q} \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a basis for the topology on $\mathbb{R}^n$. Let X^{2n+1} be as in (A). Let Y^n be the set given by

$$Y^n := \{(s_0, \ldots, s_{n-1}, r) \in \mathbb{R}^{n+1} : \text{There is some } (m_0, \ldots, m_{2n}) \text{ in } X^{2n+1}_r$$
$$\text{such that } m_{2n}|m_{2i} - m_{2i+1}s_i| < m_{2i+1} \text{ for } i = 0, \ldots, n-1\}.$$

The set Y^n satisfies our requirements.

(C) The graph of any continuous function $f : \mathbb{R}^m \to \mathbb{R}^n$ and any function $g : \mathbb{N}^m \to \mathbb{R}^n$ is a closed subset of R^{m+n}. Hence the desired result follows from (B).

□

We next explore the consequences of Proposition 6.1 in $\mathcal{R}$. Let $\mathbb{N}^*$ denote the interpretation in $\mathcal{R}$ of the $\mathscr{L}_{\text{proj}}$-formula "$x_0 \in \mathbb{Z} \wedge x_0 \geq 0$." We use the letters α, β, γ, and δ to denote elements of $\mathbb{N}^*$. We remind the reader that below "definable" means "definable in $\mathcal{R}$".

When transferred to $\mathcal{R}$, Proposition 6.1 becomes the following statement:

Proposition 6.2. *Let $n \geq 1$.*

(a) *There is a definable (without parameters) subset X^n of R^{n+1} such that $\{X^n_r : r \in R\}$ is equal to the collection of all definable subsets of $(\mathbb{N}^*)^n$.*

(b) *There is a definable (without parameters) subset Y^n of R^{n+1} such that $\{Y^n_r : r \in R\}$ is equal to the collection of all definable open subsets of R^n. Consequently, the same is true with "open" replaced by "closed."*

(c) *Let $m \geq 1$. There is a definable (without parameters) subset $Z^{m,n}$ of R^{m+n+1} such that $\{Z^{m,n}_r : r \in R\}$ is equal to the collection of all graphs of definable continuous functions from R^m to R^n. Moreover, the same is true for all definable functions from $(\mathbb{N}^*)^m$ to R^n.*

Proof. Simply let $X^n, Y^n, Z^{m,n}$ be the interpretations in $\mathcal{R}$ of the corresponding sets obtained in Proposition 6.1. □

In this first Corollary, we compare definable connectedness and definable path connectedness in $\mathcal{R}$-manifolds.

Corollary 6.3. *Let X be a definable subset of R^n. Then each p-component of X is definable. Consequently, a definably connected $\mathcal{R}$-manifold is definably path connected.*

Proof. Choose an a in X. By Proposition 6.2, there is an $\mathscr{L}_{\text{proj}}$-formula $\phi(x, a)$ that expresses the statement "There exists a definable continuous path $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = x$." This formula defines the p-component of X containing a. The consequence follows from Corollary 3.10. □

In light of Proposition 6.2, we also get a relative notion of countability.

Definition 6.4. Let A be a definable subset of R^n. The set A is called **$\mathcal{R}$-countable** if there is a definable surjection $f : \mathbb{N}^* \to A$. Similarly, A is called **$\mathcal{R}$-infinite** if there is a definable injection $f : \mathbb{N}^* \to A$. When A is not $\mathcal{R}$-infinite, it is called $\mathcal{R}$-finite.

By virtue of item (C) of Proposition 6.1, a subset A of $\mathbb{R}^n$ is countable if and only if A is $\mathbb{R}_{\text{proj}}$-countable. Similarly, A is finite if and only if A is $\mathbb{R}_{\text{proj}}$-finite.

The set $\mathbb{N}^*$ can also be used to index definable sequences.

Definition 6.5. Let J be a definable subset of $\mathbb{N}^*$. Then a definable subset X of $J \times R^n$ is called a **definable sequence** in R^n. We shall denote such a collection by $(X_\alpha)_{\alpha\in J}$ or by just $(X_\alpha)_\alpha$ with the understanding that there is a definable set $J \subseteq \mathbb{N}^*$ over which α ranges. Occasionally, we shall also denote a definable sequence by $(X(\alpha))_\alpha$. If $\widetilde{J}$ is a subset of J, then the set $X \cap (\widetilde{J} \times R)$ is called a **definable subsequence** of $(X_\alpha)_{\alpha\in J}$.

A definable sequence $(X_\alpha)_\alpha$ is said to **terminate at** β if $X_\alpha = \emptyset$ whenever $\alpha \geq \beta$. If, on the other hand, the set $\{\alpha \in J : X_\alpha \neq \emptyset\}$ is unbounded, then we say the sequence $(X_\alpha)_\alpha$ is **$\mathcal{R}$-infinite**.

We next list some expected elementary results. The proofs are all easy transfer arguments.

Proposition 6.6. *(i) If A is an $\mathcal{R}$-finite subset of R^n, then A is $\mathcal{R}$-countable.*

(ii) If A is an $\mathcal{R}$-countable subset of R^n, then A has empty interior.

(iii) If $(X_\alpha)_{\alpha\in J}$ is a definable sequence of $\mathcal{R}$-countable sets, then the union $\bigcup_{\alpha\in J} X_\alpha$ is $\mathcal{R}$-countable. □

At certain points in the proof of Theorem 1.8, it is necessary to pass to subsequences. The next lemma satisfies all of our needs of this kind.

Lemma 6.7. *Let J be an unbounded subset of $\mathbb{N}^*$. Suppose $\rho : J \to (0,\infty)$ is a definable function and that $\liminf_{\alpha\to\infty} \rho(\alpha) = 0$. Then for any $r > 0$, there is a strictly increasing definable function $\delta : \mathbb{N}^* \to J$ such that*

(i) the composite function $\rho \circ \delta : \mathbb{N}^* \to (0,\infty)$ is strictly decreasing,

(ii) for all α in $\mathbb{N}^*$, we have $r\rho(\delta(\alpha+1)) < \rho(\delta(\alpha))$, and

(iii) $\lim_{\alpha\to\infty} \rho \circ \delta(\alpha) = 0$. □

In analogy with the way we obtained $\mathbb{N}^*$, there is a set $\mathbb{Q}^*$ that we use for rationals. Namely, we let $\mathbb{Q}^*$ be the interpretation in $\mathcal{R}$ of the $\mathscr{L}_{\text{proj}}$-formula $\phi(x)$ given by

$$\exists y \in \mathbb{Z}, \exists z \in \mathbb{Z}(zx = y \wedge z \neq 0).$$

The elements of $\mathbb{Q}^*$ are called **$\mathcal{R}$-rationals**. It follows from a transfer argument that $\mathbb{Q}^*$ is $\mathcal{R}$-countable and dense in R. A subset U of R^m is called an **$\mathcal{R}$-rational box** if there are $\mathcal{R}$-rationals $r_1, \ldots, r_m$ and $s_1, \ldots, s_m$ such that $r_i < s_i$ for $i = 1, \ldots, m$ and

$$U = (r_1, s_1) \times \cdots \times (r_m, s_m).$$

In this manner, we identify a rational box with an element of $(\mathbb{Q}^*)^{2m}$. As expected, rational boxes form an $\mathcal{R}$-countable basis for the topology on R^n.

Next, we state a Proposition that replaces topological compactness for closed and bounded subsets of R^n. A definable sequence of sets is said to be **increasing** (resp. **decreasing**) if whenever α and β are in J and $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$ (resp. $X_\beta \subseteq X_\alpha$).

Proposition 6.8. *(i) If W is a definable subset of R^n that is closed and bounded, and if $(U_\alpha)_\alpha$ is an increasing sequence of open subsets such that $W \subset \bigcup_\alpha U_\alpha$, then there exists a $\gamma \in \mathbb{N}^*$ such that $W \subset U_\gamma$.*

(ii) If $(W_\alpha)_{\alpha\in J}$ is a decreasing definable sequence of closed and bounded nonempty sets, then the intersection $\bigcap_\alpha W_\alpha$ is nonempty.

□

We also get a version of the classical Baire Category Theorem. As usual, a subset Y of R^n is called **nowhere dense** if $\text{int}(\text{cl}(Y)) = \emptyset$.

Proposition 6.9 (Baire Category Theorem). *Let $(Y_\alpha)_\alpha$ be a definable sequence of sets with the property that each Y_α is nowhere dense. Then the union $\bigcup_\alpha Y_\alpha$ has empty interior.* □

Finally, we define the Hausdorff limit of a definable sequence. For two definable closed and bounded nonempty subsets V and W of R^n, the **Hausdorff Distance** $d(V,W)$ is defined by

$$d(V,W) := \max\{\inf\{d(x,V) : x \in W\}, \inf\{d(x,W) : x \in V\}\}.$$

(See (2.1) for the definitions of $d(x,V)$ and $d(x,W)$.) This yields a metric on the collection of definable nonempty closed and bounded subsets of R^n.

If it exists, the limit $\lim X_\alpha$ of a definable ($\mathcal{R}$-infinite) sequence $(X_\alpha)_\alpha$ in the induced topology is called the **Hausdorff limit of** $(X_\alpha)_\alpha$. When $\mathcal{R} = \mathbb{R}_{\text{proj}}$, it is a classical fact that a closed and bounded collection of nonempty compact subsets of $\mathbb{R}^n$ is compact in this topology. (See for example p. 279 of Munkres [22].) In particular, every uniformly bounded sequence of closed subsets of $\mathbb{R}^n$ has a convergent subsequence.

Since Corollary 6.2 allows us to definably quantify over all definable subsets of $\mathbb{N}^*$ and over all closed definable subsets of R^n, a transfer argument yields the following:

Proposition 6.10. *If $(X_\alpha)_{\alpha \in J}$ is a uniformly bounded definable $\mathcal{R}$-infinite sequence of closed nonempty subsets of R^n, then there is a closed and bounded definable subset X of R^n and a definable unbounded subset $\widetilde{J}$ of J such that X is the Hausdorff limit of the definable subsequence $(X_\alpha)_{\alpha \in \widetilde{J}}$.* □

In the course of the arguments below, we need to choose Hausdorff limits in a uniformly definable way. In the classical setting we have this:

Proposition 6.11. *Fix a bounded box B in $\mathbb{R}^n$. For $p = 1, \ldots, K$ and all i and j in $\mathbb{N}$, let $Y^p(i,j)$ be a closed subset of B. Then there are unbounded subsets I*

and J of $\mathbb{N}$ such that the following hold:

(I) *For each p and $i \in I$, the sequence $(Y^p(i,j))_{j\in J}$ converges to a Hausdorff limit $Y^p(i)$ whenever it is infinite. In case this sequence is not infinite, we set $Y^p(i) := \emptyset$.*

(II) *For each p, the sequence $(Y^p(i))_{i\in I}$ converges to a Hausdorff limit Y^p whenever it is infinite.*

Proof. This is a standard diagonal argument of the kind used in the classical Ascoli-Arzelà Theorem. (See p. 167 of Royden [24].) □

Transferring the previous Proposition yields the next one.

Proposition 6.12. *Fix a bounded box B in R^n. For each $p = 1, \dots, K$, let Y^p be a definable subset of $(\mathbb{N}^*)^2 \times B$ such that for all α and β in $\mathbb{N}^*$, the set $Y^p(\alpha,\beta) := Y^p_{(\alpha,\beta)}$ is closed. Then there are unbounded definable subsets I and J of $\mathbb{N}^*$ such that the following hold:*

(i) *For each p and $\alpha \in I$, the definable sequence $(Y^p(\alpha,\beta))_{\beta\in J}$ converges to a Hausdorff limit $Y^p(\alpha)$ if this sequence is $\mathcal{R}$-infinite. In case this sequence is not $\mathcal{R}$-infinite, we set $Y^p(\alpha) := \emptyset$.*

(ii) *For each p, the definable sequence $(Y^p(\alpha))_{\alpha\in I}$ converges to a Hausdorff limit Y^p whenever it is $\mathcal{R}$-infinite.*

Equipped with these tools, we resume our study of Pfaffian closures.

7 Relative Pfaffian closures

Let $\widetilde{\mathcal{S}}$ be an expansion of a real closed ordered field in a language $\widetilde{\mathscr{L}}$. Let $\mathcal{S}$ be an expansion of $\widetilde{\mathcal{S}}$ with the IVP. For all natural numbers i, we define languages $\mathscr{L}_i$ recursively as follows: Set $\mathscr{L}_0 = \widetilde{\mathscr{L}}$. Then $\mathscr{L}_{i+1}$ is the language obtained by adding to $\mathscr{L}_i$, for each $\mathscr{L}_i$-definable 1-form ω and each Rolle $\mathcal{S}$-leaf L of $\omega = 0$, a predicate symbol for L. Define $\mathscr{L}_{\text{Rolle}}$ to be the language

$$\mathscr{L}_{\text{Rolle}} := \bigcup_{i \in \mathbb{N}} \mathscr{L}_i,$$

and define the structure $\mathscr{P}(\widetilde{\mathcal{S}}, \mathcal{S})$ to be the expansion of $\widetilde{\mathcal{S}}$ to the language $\mathscr{L}_{\text{Rolle}}$. We call the structure $\mathscr{P}(\widetilde{\mathcal{S}}, \mathcal{S})$ the **relative Pfaffian closure of $\widetilde{\mathcal{S}}$ in $\mathcal{S}$**.

While there is no mention of o-minimality in this definition, we are for now only interested in the case where $\widetilde{\mathcal{S}}$ is o-minimal. Note for example that when $\widetilde{\mathbb{R}}$ is an o-minimal expansion of $\mathbb{R}$, it follows from Proposition 6.1 and Corollary 6.3 that $\mathscr{P}(\widetilde{\mathbb{R}}, \mathbb{R}_{\text{proj}})$ is identical to Speissegger's Pfaffian closure $\mathscr{P}(\widetilde{\mathbb{R}})$ of $\widetilde{\mathbb{R}}$.

Recall now that $\mathcal{R}$ is a model of T^{proj} with an o-minimal reduct $\widetilde{\mathcal{R}}$. Let $\widetilde{\mathscr{L}}$ be the language of $\widetilde{\mathcal{R}}$. Now any set definable in $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ is definable in the language $\mathscr{L}_i$ for some i. If we define $\mathscr{P}_i(\widetilde{\mathcal{R}}, \mathcal{R})$ to be the expansion of $\widetilde{\mathcal{R}}$ to the language $\mathscr{L}_i$, we see that

$$\mathscr{P}_{i+1}(\widetilde{\mathcal{R}}, \mathcal{R}) = \mathscr{P}_1(\mathscr{P}_i(\widetilde{\mathcal{R}}, \mathcal{R}), \mathcal{R}).$$

Thus to show Theorem 1.8, it suffices to show the following:

Theorem 7.1. *The structure $\mathscr{P}_1(\widetilde{\mathcal{R}}, \mathcal{R})$ is o-minimal.*

After having read Chapter 6, a reader might reasonably ask whether Proposition 7.1 could be obtained simply via a transfer argument. A little thought, however, reveals that such an argument requires a means to express in $\mathcal{R}$ that a given set is definable in $\widetilde{\mathcal{R}}$. Perhaps this could be done, but it would demand entirely different methods. Also, by circumventing the main obstacles, this approach is less likely to deepen our understanding of either the new or the established case. Instead, we opt to broaden an already blazed trail, following [25], yet watching for previously irrelevant pitfalls and unnoticed vistas.

We begin at the level of $\widetilde{\mathcal{R}}$-Pfaffian sets.

Definition 7.2. A subset W of R^n is a **basic $\widetilde{\mathcal{R}}$-Pfaffian set** if there are $\widetilde{\mathcal{R}}$-definable 1-forms $\omega_1, \ldots, \omega_q$ on a common open subset U of R^n, Rolle $\mathcal{R}$-leaves L_i of $\omega_i = 0$ for $i = 1, \ldots, q$, and an $\widetilde{\mathcal{R}}$-definable subset A of U such that

$$W = A \cap L_1 \cap \cdots \cap L_q.$$

An **$\widetilde{\mathcal{R}}$-Pfaffian set** is a finite union of basic $\widetilde{\mathcal{R}}$-Pfaffian sets.

Building on the methods of [18], the proof of o-minimality in [25] is axiomatic; rather than working with $\widetilde{\mathcal{R}}$-Pfaffian sets directly, the properties that make the proof work are isolated through a system of Λ-sets. In parallel with that development, we fix a system $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$ of collections Λ_n of definable subsets of R^n. (Reminder: "definable" means "definable in $\mathcal{R}$.") A subset W of R^n is called a Λ-set if $W \in \Lambda_n$ for some n. We also require that the following seven axioms be satisfied for $W \in \Lambda_n$:

(I) If a subset Z of R^n is definable in $\overline{\mathcal{R}}$, then Z is in Λ_n.

(II) If $Z \in \Lambda_n$, then the intersection $W \cap Z$ is in Λ_n.

(III) The cartesian product $W \times R$ is in Λ_{n+1}.

(IV) If π is a permutation of $\{1, \ldots, n\}$, then the set $\pi(W) := \{(x_{\pi(a)}, \ldots, x_{\pi(n)}) : x \in W\}$ is in Λ_n.

(V) There is a natural number m greater that n and a closed W' in Λ_m such that $W = \Pi_n(W')$.

(VI) If $1 \leq k \leq n$, then there is a natural number K such that for each z in R^k the fiber W_z has fewer than K components.

(VII) There is a natural number K and sets $W^1, \ldots, W^K$ in Λ_n such that $W = \bigcup_{p=1}^{K} W^p$ and each W^p is an $\mathcal{R}$-manifold in standard position. (See Definition 3.5.)

From the Λ-sets we obtain the Λ^∞-sets.

Definition 7.3. Let W be a definable subset of $R^k \times R^m \times R^l$, and let $\epsilon : (\mathbb{N}^*)^2 \to R^k$ be a definable function. A subset X of R^m is **obtained from** W if the following conditions hold:

(i) For each pair (α, β), the fiber $W_{\epsilon(\alpha,\beta)}$ is a closed and bounded subset of $R^m \times R^l$.

(ii) For each α in $\mathbb{N}^*$, the sequence $(W(\alpha, \beta))_\beta$ of subsets of R^m is decreasing, where $W(\alpha, \beta) := \Pi_m(W_{\epsilon(\alpha,\beta)})$.

(iii) The sequence $(W(\alpha))_\alpha$ is increasing, where $W(\alpha) := \bigcap_\beta W(\alpha, \beta)$.

(iv) Finally, $X = \bigcup_\alpha W(\alpha)$.

When a set X is obtained from a Λ-set W, we call X a **basic Λ^∞-set**. A **Λ^∞-set** is a finite union of basic Λ^∞-sets.

Remarks: In the case where $\mathcal{R} = \mathbb{R}_{\text{proj}}$, this definition agrees with the definition in [25]. Also, since the function ϵ is required to be definable, each Λ^∞-set is definable too.

In this terminology, the proof of Theorem 7.1 follows from three Lemmas and a Proposition, which we now state. Let $\mathcal{I}$ be the closed interval $[-1, 1]$.

Lemma 7.4. *The collection of $\widetilde{\mathcal{R}}$-Pfaffian sets satisfies Axioms (I)-(VII) in the definition of Λ-sets.*

Lemma 7.5. *Every Λ-set is a Λ^∞-set.*

Lemma 7.6. *Let W be a Λ-set. Then there is a natural number K such that every basic Λ^∞-set obtained from W has fewer than K components.*

Proposition 7.7. *The collections of Λ^∞-subsets of $\mathcal{I}^n$ (as n ranges over $\mathbb{N}$) form a structure on $\mathcal{I}$.*

The next two chapters are devoted to the lengthy proof of Proposition 7.7, and we shall prove Lemmas 7.4, 7.5, and 7.6 shortly. For now, let us assume these results and prove Theorem 7.1.

Proof of Theorem 7.1. Let $\tau_n : R^n \to (-1,1)^n$ be the $\overline{\mathcal{R}}$-definable diffeomorphism given by

$$\tau_n(x_1, ..., x_n) := \left(\frac{x_1}{\sqrt{1+x_1^2}}, ..., \frac{x_n}{\sqrt{1+x_n^2}} \right).$$

Let $\mathscr{T}_n$ be the collection of Λ^∞- subsets of $\mathcal{I}^n$, and put

$$\mathscr{I}_n := \{\tau_n^{-1}(X) : X \in \mathscr{T}_n\}.$$

For each natural number n, we take Λ_n to be the collection of $\widetilde{\mathcal{R}}$-Pfaffian subsets of R^n. Then by Lemma 7.4, Lemma 7.5, and Proposition 7.7, the system $(\mathscr{T}_n)_n$ forms a structure on $\mathcal{I}$. That this structure is o-minimal is a consequence of Lemma 7.6. It then follows that the system $(\mathscr{I}_n)_n$ forms an o-minimal structure $\mathcal{S}$ on R by virtue of the homeomorphism τ_1.

We next show that $\mathcal{S}$ expands $\widetilde{\mathcal{R}}$. Let Y be an $\widetilde{\mathcal{R}}$-definable subset of R^n. So $\tau_n(Y)$ is $\widetilde{\mathcal{R}}$-definable and hence is an $\widetilde{\mathcal{R}}$-Pfaffian subset of $\mathcal{I}^n$. By Lemma 7.5, the set $\tau_n(Y)$ is also in $\mathscr{T}_n$, which is what we needed to show.

To finish the proof it suffices to show that, for every $\widetilde{\mathcal{R}}$-definable 1-form ω on U, each Rolle $\mathcal{R}$-leaf L of $\omega = 0$ is definable in $\mathcal{S}$; for then it follows that $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ is a reduct of the o-minimal structure $\mathcal{S}$. Given ω as above, the pullback $\zeta := (\tau^{-1})^*\omega$ is a 1-form on $\tau(U)$. By Lemma 5.7, the image $\tau(L)$ is a Rolle $\mathcal{R}$-leaf of $\zeta = 0$. Hence $\tau(L)$ is a Λ_n-set and L is definable in $\mathcal{S}$. $\square$

We conclude this Chapter by proving Lemmas 7.4, 7.5, and 7.6 above (in reverse order).

Proof of Lemma 7.6. Let K be the bound obtained for W from Axiom (VI) for Λ-sets. We claim that every basic Λ^∞-set obtained from W has fewer than K components.

First off, if a subset D of R^{m+l} has fewer than K components, then the set $\Pi_m(D)$ also has fewer than K components.

Next, assume that $(D_\alpha)_\alpha$ is a decreasing sequence of closed and bounded subsets of R^m and that each D_α has fewer than K components, then the intersection $D := \bigcap_\alpha D_\alpha$ also has fewer than K components: We may assume D is nonempty. Let $U_1, ..., U_K$ be definable pairwise disjoint open subsets of R^m, and assume that the set $D \cap U_i$ is not empty for $i = 1, ..., K$. Put $B := R^m \setminus \bigcup_{i=1}^K U_i$ and $B_\alpha := B \cap D_\alpha$ for each α. We have that $(B_\alpha)_\alpha$ is a definable decreasing sequence of closed and bounded sets. Now $D_\alpha \cap U_i$ is nonempty for each α and $i = 1, \dots, k$. Thus since D_α has fewer than K components, the set B_α is nonempty for each α. It follows from Proposition 6.8 that $\bigcap_{i=1}^K B_\alpha$ is nonempty. In other words, D is not contained in the union $\bigcup_{i=1}^K U_i$.

Finally, it is easy to see that if $(D_\alpha)_\alpha$ is an increasing sequence of closed and bounded sets and if each D_α has fewer than K components, then the union $D := \bigcup_\alpha D_\alpha$ has fewer than K components: If $U_1, ..., U_K$ are any subsets of R^n with the property that $D \cap U_i$ is nonempty for $i = 1, ..., K$, then there must be an α such that $D_\alpha \cap U_i$ is nonempty for $i = 1, ..., K$. $\square$

Proof of Lemma 7.5. Suppose W is in Λ_n, and assume for the moment that W is closed. Put

$$\widetilde{W} := \{(t, x) \in (0, \infty) \times R^n : x \in W \text{ and } |x| \le t\}.$$

By Axioms (I),(II), and (III), the set $\widetilde{W}$ is in Λ_{n+1}. Observe also that the set $\widetilde{W}_\alpha$ is closed and bounded for each α in $\mathbb{N}^*$ and that $W = \bigcup_\alpha \widetilde{W}_\alpha$. This proves the Lemma for closed sets.

For a general Λ_n-set W, Axiom (V) says there is an $n' \geq n$ and a closed $\Lambda_{n'}$-set W' such that $W = \Pi_n(W')$. Since we just proved that W' is a Λ^∞-set, it follows that its projection W is a Λ^∞-set as well. □

Proof of Lemma 7.4. First note that Axioms (I) and (IV) are obvious. In addition, for the other Axioms it suffices to consider a basic Λ_n-subset $W = A \cap L_1 \cap ... \cap L_q$ of R^n, where A is $\widetilde{\mathcal{R}}$-definable and each L_i is a Rolle $\mathcal{R}$-leaf of a 1-form ω_i on a common open set U.

Axiom (II): Suppose $Z = B \cap L_{q+1} \cap ... \cap L_p$, where B is an $\widetilde{\mathcal{R}}$-definable subset of R^n and each L_i is a Rolle $\mathcal{R}$-leaf of $\omega_i = 0$ for some $\widetilde{\mathcal{R}}$-definable 1-forms $\omega_{q+1}, ..., \omega_p$ defined on an open set U_1. We claim that $W \cap Z$ is $\widetilde{\mathcal{R}}$-Pfaffian. By Theorem 1.7, the set $L_i \cap U \cap U_1$ is a finite union of Rolle $\mathcal{R}$-leaves of $\omega_i|_{U \cap U_1} = 0$ for each $i = 1, \dots, p$. It then follows that $W \cap Z$ is the intersection of $A \cap B$ with a boolean combination of Rolle R-leaves of $\omega_i|_{U \cap U_1} = 0$ for $i = 1, \dots, q$.

Axiom (III): Put $\widetilde{A} := A \times R$ and let ω_i' be the 1-form on $U \times R$ determined by the vector field $F_i' = (F_i, 0)$. Then $\widetilde{L}_i := L_i \times R$ is a Rolle $\mathcal{R}$-leaf of $\omega_i' = 0$ and $W \times R = \widetilde{A} \cap \widetilde{L}_1 \cap \cdots \cap \widetilde{L}_q$.

Axiom (V): By Corollary 4.7, there is an $n' \geq n$ and a closed $\widetilde{\mathcal{R}}$-definable subset B of $R^{n'}$ such that $A = \Pi_n(B)$. Now let $U' = U \times R^{(n'-n)}$ and let ω_i' be the 1-form on U' determined by the vector field $F_i'(x) = (F(x), 0, \dots, 0)$ for $i = 1, \dots, q$. Also put $L_i' := L_i \times R^{(n'-n)}$ and $W' := B \cap L_1' \cap \cdots \cap L_q'$. Then each L_i' is a Rolle $\mathcal{R}$-leaf of $\omega_i' = 0$, the set W' is closed, and $W = \Pi_n(W')$.

Axiom (VI): Let $j < n$. Here we show that there is a natural number K such that W_a has at most K components for any a in R^j. For $i = 1, \dots, j$, let ω_{q+i} be the 1-form on U determined by the constant vector field $F_{q+i}(x) = e_i$. (Recall that e_i is the i-th element in the standard basis for R^n.) Then by Theorem 1.7, there is a natural number K' such that whenever L_i is a Rolle $\mathcal{R}$-leaf of $\omega_i = 0$ for $i = 1, \ldots, q+j$ the set $A \cap L_1 \cap \cdots \cap L_{q+j}$ has fewer than K' components.

Now for each c in R, each component of the set $\{x \in U : x_i = c\}$ is a Rolle $\mathcal{R}$-leaf of $\omega_{q+i} = 0$. Thus by Theorem 1.7 again, there is a natural number K'' such that the set $\{x \in U : x_1 = a_1, \ldots, x_j = a_j\}$ has fewer than K'' components for all $a \in R^j$. Thus for any a in R^j, the set W_a has fewer than $K := K' \cdot K''$ components.

Axiom (VII): This Axiom is immediate from the following claim:

Claim. There is a decomposition $\mathcal{P}$ of R^n into C^1-cells such that $\mathcal{P}$ is compatible with both A and U and satisfies the following property: Whenever a cell N in $\mathcal{P}$ is a subset of A and L_i is a Rolle $\mathcal{R}$-leaf of $\omega_i = 0$ for $i = 1, \ldots, q$, then $N \cap L_1 \cap \cdots \cap L_q$ is an $\mathcal{R}$-manifold in standard position.

As in the proof of Axiom (VI), let ω_i be the 1-form determined by the constant vector field $F_{q+i}(x) = e_i$ for $i = \{1, \ldots, n\}$. Then let $\mathcal{P}$ be the decomposition given by Lemma 4.11 applied to the family $\Omega = (\omega_1, \ldots, \omega_{q+n})$. It then follows from Lemma 5.9 that the set $M := N \cap L_1 \cap \cdots \cap L_q$ is an $\mathcal{R}$-manifold for each N in $\mathcal{P}$.

To see that $N \cap L_1 \cap \cdots \cap L_q$ is in standard position, take a strictly increasing function $\iota : \{1, \ldots, k\} \to \{1, \ldots, n\}$. If we set

$$J := \{1, \ldots, q\} \cup \{q + \iota(i) : i = 1, \ldots, k\},$$

then Lemma 4.11 tells us there is a subset J' of J such that $\Omega_{J'}$ forms a basis for Ω_J along N.

Now by Example 3.3, for any a in M we have

$$\mathrm{rank}_a(\Pi_\iota|_M) = \dim(\Pi_\iota(T_a M)).$$

Moreover,

$$\begin{aligned}
\ker(\Pi_\iota|_{T_aM}) &= T_aM \cap \ker \Pi_\iota \\
&= T_aN \cap \textstyle\bigcap_{i=1}^{q} \ker(\omega_i(a)) \cap \bigcap_{i=1}^{k} e_{\iota(i)}^{\perp} \\
&= T_aN \cap \textstyle\bigcap_{i\in J'} \ker(\omega_i(a)).
\end{aligned}$$

Thus the $\text{rank}_a(\Pi_\iota|_M) = \dim(M) - \dim(N) + |J'|$ for all a in M. □

8 Closure properties for Λ^∞-sets

This Chapter and the next are devoted entirely to the proof of Proposition 7.7, which says that the Λ^∞-subsets of $\mathcal{I}^n$ form a model-theoretic structure (as n varies). By far, the most difficult part of the proof is to show that the complement of a Λ^∞-set is itself a Λ^∞-set. Let us warm up with the easier tasks.

Proposition 8.1. *The collection of Λ^∞-sets is closed under taking finite unions, finite intersections, cartesian products, permutations of coordinates, coordinate projections, and fibers. Moreover, any $\overline{\mathcal{R}}$-definable set is a Λ^∞-set.*

Proof. That the collection of Λ^∞-sets is closed under taking finite unions, cartesian products, permutations of coordinates, coordinate projections, and fibers is immediate from the definitions. For example, if X is a Λ^∞-set obtained from W via a definable function $\epsilon : (\mathbb{N}^*)^2 \to R^k$ and if a is in R, then the fiber X_a is obtained from W via the definable function $\widetilde{\epsilon}(\alpha, \beta) := (\epsilon(\alpha, \beta), a)$. Moreover, the fact that an $\overline{\mathcal{R}}$-definable set is a Λ^∞-set is a consequence of Lemma 7.5 and Axiom (I) for Λ-sets.

For finite intersections, it suffices to consider basic Λ^∞-sets. Suppose that X and X' are basic Λ^∞-sets obtained from subsets W and W' of R^n and $R^{n'}$ respectively. Then $X \times X'$ is obtained from the Λ-set

$$\widetilde{W} := \{(\theta, \theta', y, y', z, z') \in R^{n+n'} : \ (\theta, y, z) \in W \text{ and } (\theta', y', z') \in W'\}.$$

Let Δ and $\widetilde{\Delta}$ be the sets

$$\begin{aligned} \Delta &:= \{(y, y') \in R^{2m} : y = y'\} \quad \text{and} \\ \widetilde{\Delta} &:= \{(\theta, \theta', y, y', z, z') \in R^{n+n'} : y = y'\}. \end{aligned}$$

Then $(X \times X') \cap \Delta$ is obtained from $\widetilde{W} \cap \widetilde{\Delta}$. Since $X \cap X' = \Pi_m((X \times X') \cap \Delta)$, we see that $X \cap X'$ is also obtained from $\widetilde{W} \cap \widetilde{\Delta}$; that is to say, $X \cap X'$ is a Λ^∞-set. □

Remarks: The proof above shows that if X is obtained from W, then the fiber X_a is also obtained from W. Hence by Lemma 7.6, there is a K independent of a, such that X_a has fewer than K components. In addition, if X' is obtained from W', then $X \cap X'$ is obtained from the set $\widetilde{W} \cap \widetilde{\Delta}$, which depends on W and W' but not on X nor X'.

It remains to show that the Λ^∞-subsets of $\mathcal{I}^m$ are closed under taking relative complements. We restate this as a Proposition:

Proposition 8.2. *If a subset X of $\mathcal{I}^m$ is a Λ^∞-set, then so is $\mathcal{I}^m \setminus X$.*

To prove this, we collect a few more helpful facts—for instance, that bounded Λ^∞-sets are closed under taking topological closures.

Proposition 8.3. *If X is a bounded Λ^∞-set, then* $\mathrm{cl}(X)$ *is a Λ^∞-set.*

Proof. Again we may assume that X is a basic Λ^∞-set. So suppose X is bounded and obtained from a Λ-set W. We adopt the notations from Definition 7.3, and put $\widetilde{n} = k + 1 + m + m + l$. Now let $\widetilde{W}$ be the set in $\Lambda_{\widetilde{n}}$ defined by

$$\widetilde{W} := \{(\theta, t, y, y', z) : \ (\theta, y', z) \in W \text{ and } d(y, y') \leq t\}.$$

Since $W_{\epsilon(\alpha,\beta)}$ is closed and bounded for each α and β, the fiber $\widetilde{W}_{\epsilon(\alpha,\beta),t}$ is also closed and bounded. We also see that

$$\Pi_m(\widetilde{W}_{\epsilon(\alpha,\beta),t}) = S(W(\alpha, \beta), t)$$

by unravelling definitions as follows. (See the conventions in Chapter 1 for the definition of $S(W(\alpha, \beta), t)$.)

$$\begin{aligned} y \in S(W(\alpha,\beta),t) &\Leftrightarrow \exists y' \in W(\alpha,\beta) \text{ such that } d(y,y') \leq t \\ &\Leftrightarrow \exists (y',z) \in W_{\epsilon(\alpha,\beta)} \text{ such that } d(y,y') \leq t \\ &\Leftrightarrow y \in \Pi_m(\widetilde{W}_{\epsilon(\alpha,\beta),t}). \end{aligned}$$

We prove the existence of a definable function $\widetilde{\theta} : \mathbb{N}^* \to R^{k+1}$ such that $(\Pi_m(\widetilde{W}_{\widetilde{\theta}(\alpha)}))_\alpha$ is decreasing and

$$\bigcap_\alpha \Pi_m\left(\widetilde{W}_{\widetilde{\theta}(\alpha)}\right) = \mathrm{cl}(X).$$

Define the non-increasing sequences:

$$\begin{aligned} \rho(\alpha) &:= \inf\{\rho \in (0,\infty) : \mathrm{cl}(X) \subseteq T(W(\alpha),\rho)\} \text{ and} \\ \rho(\alpha,\beta) &:= \inf\{\rho \in (0,\infty) : W(\alpha,\beta) \subseteq T(W(\alpha),\rho)\}. \end{aligned}$$

Since each of the sets $\mathrm{cl}(X)$, $W(\alpha)$, and $W(\alpha,\beta)$ is closed and bounded, it follows that for all α and β,

$$\begin{aligned} \mathrm{cl}(X) &\subseteq S(W(\alpha),\rho(\alpha)) \text{ and} \\ W(\alpha,\beta) &\subseteq S(W(\alpha),\rho(\alpha,\beta)) \end{aligned}$$

We claim that $\lim_{\alpha\to\infty}\rho(\alpha) = 0$ and for each α that $\lim_{\beta\to\infty}\rho(\alpha,\beta) = 0$: Fix $\rho > 0$ and notice that

$$\begin{aligned} \mathrm{cl}(X) &\subset \textstyle\bigcup_\alpha T(W(\alpha),\rho) \text{ and that} \\ \textstyle\bigcap_\beta W(\alpha,\beta) &\subset T(W(\alpha),\rho) \text{ for all } \alpha. \end{aligned}$$

Proposition 6.8 then implies that there is a γ in $\mathbb{N}^*$ such that

$$\mathrm{cl}(X) \subset T(W(\gamma),\rho)).$$

Similarly, for each α there is a γ_α such that

$$W(\alpha,\gamma_\alpha) \subset T(W(\alpha),\rho).$$

In other words, $\rho(\gamma) < \rho$ and $\rho(\alpha,\gamma_\alpha) < \rho$.

Now if there is an α such that $\rho(\alpha) = 0$, then $\mathrm{cl}(X) = W(\alpha)$. So we may assume $\rho(\alpha) > 0$ for all α. In this case, for each α there is a least β_α such that

$\rho(\alpha, \beta_\alpha) < \rho(\alpha)$. Moreover, by Lemma 6.7 we may assume that $3\rho(\alpha+1) < \rho(\alpha)$ for each α. Now since $W(\alpha) \subseteq \mathrm{cl}(X)$ and $W(\alpha) \subseteq W(\alpha, \beta_\alpha)$ for all α, we have

$$S(\mathrm{cl}(X), 3\rho(\alpha+1)) \subseteq S(\mathrm{cl}(X), \rho(\alpha)) \subseteq S(W(\alpha, \beta_\alpha), 2\rho(\alpha)) \subseteq S(\mathrm{cl}(X), 3\rho(\alpha)).$$

Therefore, the sequence $(\Pi_m(\widetilde{W}_{\widetilde{\theta}(\alpha)}))_\alpha$ is decreasing and

$$\mathrm{cl}(X) = \bigcap_\alpha S(W(\alpha, \beta_\alpha), 2\rho(\alpha)) = \bigcap_\alpha \Pi_m(\widetilde{W}_{\widetilde{\theta}(\alpha)})$$

where $\widetilde{\theta}(\alpha) = (\alpha, \beta_\alpha, 2\rho(\alpha))$. This finishes the proof when X is bounded. □

In addition, we need a Lemma and another Proposition. The Lemma makes use of our version of the Baire Category Theorem (Proposition 6.9).

Lemma 8.4. *(i) If $X^1, ..., X^q$ are Λ^∞-sets with empty interior, then their union $\bigcup_{i=1}^q X^i$ has empty interior.*

(ii) Suppose $m > 1$ and that X is a Λ^∞-set contained in R^n. Then X has empty interior if and only if the set $\mathcal{K} := \{a \in R : \mathrm{int}(X_a) \neq \emptyset\}$ has empty interior.

Proof. (i) For each i, there is a definable sequence of closed sets $W^i(\alpha)$ such that $X^i = \bigcup_\alpha W^i(\alpha)$. Consequently, each $W^i(\alpha)$ is nowhere dense. Thus the set

$$\bigcup_{i=1}^q X^i = \bigcup_{\alpha \in \mathbb{N}^*} (W^1(\alpha) \cup \cdots \cup W^q(\alpha))$$

has empty interior by the Baire Category Theorem.

(ii) The right to left direction is trivial, so suppose $\mathcal{K}$ has nonempty interior. Since X is the union of closed sets $W(\alpha)$, it suffices to show that $W(\alpha)$ has interior for some α. We also know that the fiber X_a is the union of the closed fibers $(W(\alpha))_a$ for each a in R. It follows from the Baire Category Theorem, that $\mathcal{K} \subseteq \bigcup_\alpha \mathcal{K}(\alpha)$ where

$$\mathcal{K}(\alpha) := \{a \in R : \mathrm{int}(W(\alpha))_a \neq \emptyset\}.$$

At the same time

$$\mathcal{K}(\alpha) = \bigcup_U \{a \in R : U \subset (W(\alpha))_a\}$$

where the union is taken over all $\mathcal{R}$-rational boxes U. Using the Baire Category Theorem again, we see that for some α and U the set

$$\{a \in R : U \subset (W(\alpha))_a\}$$

is dense in an interval I. Now since $W(\alpha)$ is closed, the box $I \times U$ is contained in $W(\alpha)$ and hence in X. Therefore, X has nonempty interior.

□

The following Proposition conceals much of the difficulty of Proposition 8.2; we postpone its proof until the next Chapter.

Proposition 8.5. *Suppose a bounded subset X of R^m is a Λ^∞-set. Then there is a closed Λ^∞-set Y with empty interior such that* $\mathrm{bd}(X) \subseteq Y$.

Let us assume Proposition 8.5 for what remains of this Chapter. We finish the proof of Proposition 8.2 after a quick Corollary.

Corollary 8.6. *Suppose a bounded subset X of R^m is a Λ^∞-set for some $m > 1$. Then the set*

$$B := \{a \in R^{m-1} : \mathrm{cl}(X_a) \neq \mathrm{cl}(X)_a\}$$

has empty interior.

Proof. For each a in B there is an open interval U with $\mathcal{R}$-rational endpoints such that $\mathrm{cl}(X_a) \cap U$ is empty and $\mathrm{cl}(X)_a \cap U$ is nonempty. Hence $B = \bigcup_U B_U$ where U ranges over all such intervals and

$$B_U := \{a \in R^{m-1} : \mathrm{cl}(X_a) \cap U = \emptyset \text{ and } \mathrm{cl}(X)_a \cap U \neq \emptyset\}.$$

Also, each B_U is contained in the frontier of the bounded Λ^∞-set $\Pi_{m-1}(X \cap (R^{m-1} \times U))$. Thus by Proposition 8.5, the set B_U is contained in a closed set Y with empty interior. It follows that $\mathrm{cl}(B_U)$ is nowhere dense. The Baire Category Theorem now implies that the union $\bigcup_U \mathrm{cl}(B_U)$ has empty interior. Therefore, B has empty interior. □

Now we come to the proof of Proposition 8.2. In fact, the proof from [25] goes through as is. We reproduce it with cosmetic changes.

Proof of Proposition 8.2. Suppose a subset X of $\mathcal{I}^m$ is a Λ^∞-set. We prove the following four statements for X by induction on m.

- $(\mathrm{I})_m$ If $\mathrm{int}(X) = \emptyset$, then X can be partitioned into finitely many Λ^∞-sets $G_1, \ldots, G_k$, in such a way that for each $i \in \{1, \ldots, k\}$ there is a permutation π_i of $\{1, \ldots, m\}$ such that $\pi_i(G_i)$ is the graph of a continuous function $f_i : \Pi_{m-1}(\pi_i(G_i)) \to R$. (In case $m = 1$, this simply means that X is a singleton $\{r\}$.)
- $(\mathrm{II})_m$ The components of X are Λ^∞-sets.
- $(\mathrm{III})_m$ The complement $\mathcal{I}^m \backslash X$ is a Λ^∞-set.
- $(\mathrm{IV})_m$ The components of $\mathcal{I}^m \backslash X$ are Λ^∞-sets.

Notice that if X is the disjoint union of two Λ^∞-sets Y_1 and Y_2, and if any of $(\mathrm{I})_m$-$(\mathrm{III})_m$ holds both with Y_1 in place of X and with Y_2 in place of X, then the corresponding $(\mathrm{I})_m$-$(\mathrm{III})_m$ holds. However, we must exercise more care with $(\mathrm{IV})_m$. Moreover, whenever any of $(\mathrm{I})_m$-$(\mathrm{IV})_m$ holds, the corresponding $(\mathrm{I})_{m+1}$-$(\mathrm{IV})_{m+1}$ holds with $X \times \mathcal{I}$ in place of X.

The case $m = 1$ is easy: Components in R must be convex. Since X has only finitely many components by Lemma 7.6, each component is also definable by Proposition 2.6. Consequently, each component of X is either a point or an interval.

Now let $m > 1$, and assume that $(\mathrm{I})_k$-$(\mathrm{IV})_k$ hold for all $k < m$.

Claim. Suppose that there is a Λ^∞-set $Z \subseteq \mathcal{I}^m$ with empty interior such that $X \subseteq Z$ and $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with Z in place of X. Then $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold.

To establish the claim, take $G_1, \dots, G_k$ as in $(\mathrm{I})_m$ with Z in place of X. Then $\pi_i(G \cap X)$ is the graph of the continuous function f_i restricted to the domain $\Pi_{m-1}(\pi_i(G_i \cap X))$. This shows $(\mathrm{I})_m$.

Now X is contained in the finite disjoint union $\bigcup_{i=1,\dots,k} G_i$, and

$$\mathcal{I}^m \backslash X = (\mathcal{I}^m \backslash Z) \cup \bigcup_{i=1}^{k} G_i \backslash X.$$

Hence for $(\mathrm{II})_m$-$(\mathrm{IV})_m$, it is enough to show that the sets $G_i \backslash X$ and the components of both $G_i \backslash X$ and $G_i \cap X$ are Λ^∞-sets. So fix an $i \in \{1, \dots, k\}$, and without loss of generality assume that the permutation π_i is the identity. Now by Proposition 8.1, the set

$$\Pi_{m-1}(G_i \cap X) = \Pi_{m-1}(G_i) \cap \Pi_{m-1}(X)$$

is a Λ^∞-set. Moreover, $(\mathrm{II})_{m-1}$ tells us that the set

$$\Pi_{m-1}(G_i \backslash X) = \Pi_{m-1}(G_1) \cap (\mathcal{I}^{m-1} \backslash \Pi_{m-1}(X))$$

is a Λ^∞-set, and that the components of both $\Pi_{m-1}(G_i \cap X)$ and $\Pi_{m-1}(G_i \backslash X)$ are Λ^∞-sets. Then it follows that

$$G_i \backslash X = [\Pi_{m-1}(G_i \backslash X) \times \mathcal{I}] \cap G_i$$

is a Λ^∞-set. Similarly if C a component of $G_i \cap X$ (respectively $G_i \backslash X$), then $\Pi_{m-1}(C)$ is a component of $\Pi_{m-1}(G_1 \cap X)$ (respectively $\Pi_{m-1}(G_i \backslash X)$) and hence a Λ^∞-set. We conclude that

$$C = [\Pi_{m-1}(C) \times \mathcal{I}] \cap G_i$$

is a Λ^∞-set. This finishes the claim.

We now divide our task into two Cases:

Case 1: X has empty interior. For $i \in \mathbb{N}$, define the sets

$$\begin{aligned} C_i &:= \{a \in \mathcal{I}^{m-1} : |X_a| \geq i\} \text{ and} \\ D_i &:= \{a \in \mathcal{I}^{m-1} : |X_a| = i\}. \end{aligned}$$

Using that the collection of Λ^∞-sets is closed under projections, cartesian products, and intersections with $\overline{\mathcal{R}}$-definable sets, it is routine to see that each C_i is a Λ^∞-set. Additionally, it follows from $(\mathrm{III})_{m-1}$ that $D_i = C_i \backslash C_{i+1}$ is a Λ^∞-set.

By the first remark following Proposition 8.1, there is a number K such that the fiber X_a contains an interval whenever $|X_a| \geq K$. Thus we have that $C_i = C_K$ for all $i \geq K$. By $(\mathrm{III})_{m-1}$ again, the set $\mathcal{I}^{m-1} \backslash C_K$ is a Λ^∞-set, and it follows that the sets

$$\begin{aligned} X_1 &:= X \cap [(\mathcal{I}^{m-1} \backslash C_K) \times \mathcal{I}] \text{ and} \\ X_2 &:= X \cap (C_K \times \mathcal{I}) \end{aligned}$$

are also Λ^∞-sets. To finish this Case, it suffices to show that $(\mathrm{I})_m$-$(\mathrm{III})_m$ hold with X_1 and with X_2 in place of X and that $(\mathrm{II})_m$ also holds with the (then Λ^∞-)sets

$$\begin{aligned} X_3 &:= [(\mathcal{I}^{m-1} \backslash C_K) \times \mathcal{I}] \backslash X_1 \text{ and} \\ X_4 &:= (C_K \times \mathcal{I}) \backslash X_2. \end{aligned}$$

in place of X. (This suffices since $X = X_1 \cup X_2$ and $\mathcal{I}^m \backslash X = X_3 \cup X_4$.)

For $1 \leq j \leq i < K$, begin by setting

$$X_{i,j} := \{(a, y) \in D_i \times \mathcal{I} : y \text{ is the } j\text{-th element of } (X_1)_a\}.$$

It is again routine to verify that $X_{i,j}$ is a Λ^∞-set. If we now put

$$S_{i,j} := \{a \in D_i : |(\mathrm{cl}(X_{i,j}))_a| \geq 2\},$$

then by Proposition 8.3, each $S_{i,j}$ is also a Λ^∞-set. Notice as well that

$$S_{i,j} \subseteq \{a \in \mathcal{I}^{m-1} : \mathrm{cl}((X_{i,j})_a) \neq (\mathrm{cl}(X_{i,j}))_a\},$$

and so each $S_{i,j}$ has empty interior by Corollary 8.6. Now let S be the Λ^∞-set given by

$$S := \bigcup_{1\leq j\leq i\leq N} S_{i,j},$$

and observe that Lemma 8.4 (i), tells us that S has empty interior.

By our inductive assumption, the statements $(\mathrm{I})_{m-1}$-$(\mathrm{IV})_{m-1}$ hold for S, and consequently $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold for $S \times \mathcal{I}$. By the Claim, we conclude that $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with the set $X_1 \cap (S \times \mathcal{I})$ in place of X. On the other hand, $(\mathrm{I})_m$ also holds with $X_1 \backslash (S \times \mathcal{I})$ in place of X: for each i and j, the set $X_{i,j} \cap [(D_i \backslash S) \times \mathcal{I}]$ is the graph of a continuous function $f_{i,j} : D_i \backslash S \to R$, which is a Λ^∞-set by $(\mathrm{III})_{m-1}$. Thus $(\mathrm{I})_m$ holds with X_1 in place of X.

Now using $(\mathrm{II})_{m-1}$ and $(\mathrm{III})_{m-1}$, the components of the sets $D_i \backslash S$ and their complements $(I^{m-1} \backslash D_i) \cup S$ are all Λ^∞-sets. Note that each component of $X_1 \backslash (S \times \mathcal{I})$ is a finite disjoint union of sets of the form $X_{i,j} \cap (C \times \mathcal{I})$, where C is a component of $D_i \backslash S$. We show that $(\mathrm{II})_m$ and $(\mathrm{III})_m$ hold with these sets in place of X. Now $(\mathrm{II})_m$ is clear since $X_{i,j} \cap (C \times \mathcal{I})$ is equal to the continuous function $\Gamma(f_{i,j}|_C)$ and hence is definably connected. Notice also that the complement $\mathcal{I}^m \backslash [X_{i,j} \cap (C \times \mathcal{I})]$ is the union of the three Λ^∞-sets

$$(f_{i,j}|_C, \infty), (-\infty, f_{i,j}|_C) \text{ and } (\mathcal{I}^{m-1} \backslash C) \times \mathcal{I}.$$

The first two of these sets are definably connected, and the components of the last set are Λ^∞-sets. It follows that $(\mathrm{II})_m$ and $(\mathrm{III})_m$ hold with X_1 in place of X.

At this point we conclude that X_3 is a Λ^∞-set, and we next show that $(\mathrm{II})_m$ holds with X_3 in place of X. Now $(\mathrm{II})_m$ holds with $X_3 \cap (S \times \mathcal{I})$ in place of X, since $S \times \mathcal{I}$ has empty interior and $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold for $S \times \mathcal{I}$. Furthermore, each component of $X_3 \backslash (S \times \mathcal{I})$ is the union of definably connected sets of the forms

$$(-\infty, f_{i,1}|_C), (f_{i,j}|_C, f_{i,j+1}|_C), (f_{i,i}|_C, \infty)$$

where $1 \leq j \leq i < K$ and C ranges over the components of $D_i \backslash S$. Since these are all Λ^∞-sets, we see that $(\mathrm{II})_m$ holds with $X_3 \backslash (S \times \mathcal{I})$ in place of X. Hence $(\mathrm{II})_m$ holds with X_3 in place of X.

We turn our attention to X_2 and X_4. Note that by our inductive assumption $(\mathrm{I})_{m-1}$-$(\mathrm{IV})_{m-1}$ hold with C_K in place of X. Consequently $(\mathrm{I})_m$-$(\mathrm{IV})_m$ also hold with $C_K \times \mathcal{I}$ in place X. Observe that $C_K = \Pi_{m-1}(X_2)$ and that for each a in C_K the fiber $(X_2)_a$ contains an interval. Lemma 8.4 (ii) then tells us that C_K has empty interior so that $C_K \times \mathcal{I}$ has empty interior as well. Then since X_2 is a subset of $C_K \times \mathcal{I}$, we apply the Claim again to conclude that $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with X_2 in place of X. We know now that X_4 is a Λ^∞-set, and since X_4 is also a subset of $C_K \times \mathcal{I}$, the Claim implies that $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with X_4 in place of X. This finishes the first Case.

Case 2: X has nonempty interior. We need to show $(\mathrm{II})_m$-$(\mathrm{IV})_m$. Use Proposition 8.5 to find a closed Λ^∞-set Y with empty interior such that $\mathrm{bd}(X) \subseteq Y$. We have already shown in the first Case that $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with Y in place of X. Thus by the Claim, the statements $(\mathrm{I})_m$-$(\mathrm{IV})_m$ hold with $X \cap Y$ and $Y \backslash X$ in place of X. Now suppose C is a component of $\mathcal{I}^m \backslash Y$. Then since $\mathrm{bd}(X) \subseteq Y$, we have $C = (\mathrm{int}(X) \cap C) \cup ((\mathcal{I}^m \backslash \mathrm{cl}(X)) \cap C)$. Thus (since C is definably connected) either $X \cap C = \emptyset$ or $C \subseteq X$. It follows that $(\mathrm{II})_m$ holds since each component of X is a union of components of $X \cap Y$ and components of $\mathcal{I}^m \backslash Y$. Similarly $(\mathrm{IV})_m$ holds since each component of $\mathcal{I}^m \backslash X$ is a union of components of $\mathcal{I}^m \backslash Y$ and components of $Y \backslash X$. Since there are only finitely many such components and they are all Λ^∞-sets, it follows that $(\mathrm{III})_m$ holds as well. This completes the proof. $\square$

9 Approximating the boundary of a Λ^∞-set

In this Chapter, we prove Proposition 8.5. We start modestly with a version of Corollary 8.6.

Lemma 9.1. *Let $m > 1$. Suppose a bounded subset X of R^m is a Λ^∞-set. Then the set*

$$B := \{a \in R : \mathrm{cl}(X_a) \neq \mathrm{cl}(X)_a\}$$

is $\mathcal{R}$-countable.

Proof. For each a in B there is an $\mathcal{R}$-rational box $U \subseteq R^{m-1}$ such that $\mathrm{cl}(X_a) \cap U$ is empty and $\mathrm{cl}(X)_a \cap U$ is nonempty. Hence $B = \bigcup_U B_U$, where U ranges over all $\mathcal{R}$-rational boxes in R^{m-1} and

$$B_U := \{a \in R : \mathrm{cl}(X_a) \cap U = \emptyset \text{ and } \mathrm{cl}(X)_a \cap U \neq \emptyset\}.$$

For each U, the set B_U is contained in the frontier of the Λ^∞-set $\Pi_1(X \cap (R \times U))$ and is therefore finite by Lemma 7.6. Corollary 6.6 (iii) tells us that the set B is $\mathcal{R}$-countable. □

Lemma 9.2. *Let $m > 1$, let W be a Λ-set, and let $(Y(\alpha))_\alpha$ be a definable sequence of Λ^∞-sets in R^m obtained from W. Suppose that the sequence $(Y(\alpha))_\alpha$ converges to a Hausdorff limit Y. Then the set*

$$D := \{a \in R : Y_a \neq \emptyset \text{ and } (Y(\alpha)_a)_\alpha \text{ does not converge to } Y_a\}$$

is $\mathcal{R}$-countable.

Proof. First notice that $D := \bigcup_{U,\gamma} D_{U,\gamma}$ where U is an $\mathcal{R}$-rational box in R^{m-1} and

$$D_{U,\gamma} := \{a \in R : Y_a \cap U \neq \emptyset \text{ and } Y(\alpha)_a \cap U = \emptyset \text{ for all } \alpha \geq \gamma\}.$$

Thus it suffices to show that each $D_{U,\gamma}$ is $\mathcal{R}$-countable. Fix U and γ. By Lemma 7.6, there is a K such that $Y(\alpha)\cap(R\times U)$ has fewer than K components for all α. We show that $|D_{U,\gamma}|<2K-1$.

For $k=1,\ldots,2K-1$, assume for a contradiction that we can choose distinct a_k in $D_{U,\gamma}$. Let $s>0$ be such that the intervals $I_k:=(a_k-s,a_k+s)$ are pairwise disjoint. Then choose $\alpha\geq\gamma$ so large that $Y(\alpha)\cap(I_k\times U)\neq\emptyset$ for each k. It follows that for each component C of $Y(\alpha)\cap(R\times U)$ there are at most two numbers k such that $Y(\alpha)\cap(I_k\times U)$ meets C. This means that the set $Y(\alpha)\cap(R\times U)$ has at least K components, a contradiction. □

With these Lemmas at our disposal, we prove Proposition 8.5.

Proof. We may assume that X is a nonempty basic Λ^∞-set obtained from a subset W of R^n. Since X is bounded, there is an $r>0$ such that W is a subset of $R^k\times[-r,r]^m\times R^l$. The set Y that we are looking for is obtained with Hausdorff limits as follows:

Using Axiom (VII) for Λ-sets, we find $W^1,\ldots,W^K$ in Λ_n such that $W=\bigcup_{p=1}^K W^p$ and each W^p is an $\mathcal{R}$-manifold in standard position. In particular, each nonempty fiber W^p_ϵ with ϵ in R^k is an $\mathcal{R}$-manifold in standard position by Corollary 3.7. Moreover, there is a natural number d_p that is independent of ϵ such that whenever W^p_ϵ is nonempty the map $\Pi_m|_{W^p_\epsilon}$ has constant rank d_p.

For each α,β, and $p=1,\ldots,K$, we put $Y^p(\alpha,\beta):=\Pi_m(\mathrm{cl}(W^p_{\epsilon(\alpha,\beta)}))$. Then we apply Proposition 6.12 to get unbounded definable subsets I and J of $\mathbb{N}^*$ such that the following hold:

(i) For each p and $\alpha\in I$, the sequence $(Y^p(\alpha,\beta))_{\beta\in J}$ converges to the Hausdorff limit $Y^p(\alpha)$ if this sequence is $\mathcal{R}$-infinite. Otherwise we put $Y^p(\alpha)=\emptyset$.

(ii) For each p, the sequence $(Y^p(\alpha))_{\alpha\in I}$ converges to a Hausdorff limit Y^p if this sequence is $\mathcal{R}$-infinite. Otherwise put $Y^p=\emptyset$.

Finally, we set $Y := \bigcup_{p \in S} Y^p$ where

$$S := \{p \in \{1, \dots, K\} : d_p < m\}.$$

We shall call Y the **(I,J)-approximation of** $\mathrm{bd}(X)$ whenever Y is obtained in this way, since Y is determined uniquely by I and J (and the $\mathcal{R}$-manifolds $W^1, \dots, W^K$).

Now for each α in I, each β in J, and each p in S it is convenient to define

$$\rho_p(\alpha, \beta) := \begin{cases} d(Y^p(\alpha,\beta), Y^p(\alpha)) & \text{if } Y^p(\alpha) \neq \emptyset \text{ and } Y^p(\alpha,\beta) \neq \emptyset \\ 0 & \text{if } Y^p(\alpha) = \emptyset \text{ and } Y^p(\alpha,\beta) = \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

$$\rho_p(\alpha) := \begin{cases} d(Y^p(\alpha), Y^p) & \text{if } Y^p(\alpha) \neq \emptyset \text{ and } Y^p \neq \emptyset \\ 0 & \text{if } Y^p(\alpha) = \emptyset \text{ and } Y^p = \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Notice that for each $p = 1, \dots, K$, we have $\liminf_{\alpha \in I} \rho_p(\alpha) = 0$. Moreover, for each $p = 1 \dots, K$ and α in I, we have $\liminf_{\beta \in J} \rho_p(\alpha, \beta) = 0$.

We next show that Y satisfies the conditions of the Proposition 8.5 through three claims.

Claim (1). For each p in S, there are a natural number $n(p) \geq n$ and a $\Lambda_{n(p)}$-set $\widetilde{W}^p$ such that the sets $Y^p(\alpha)$ and Y^p are basic Λ^∞-sets obtained from $\widetilde{W}^p$. In particular, Y is a Λ^∞-set.

Fix p in S. To slightly simplify notation, we write $\rho(\alpha)$ for $\rho_p(\alpha)$ and $\rho(\alpha, \beta)$ for $\rho_p(\alpha, \beta)$. By Axiom (V), there are a number $l' \geq l$ and a closed set V in $\Lambda_{k+m+l'}$ such that $W^p = \Pi_{k+m+l}(V)$. Note that since $W^p_{\epsilon(\alpha,\beta)}$ is bounded, for each α and β we have

$$Y^p(\alpha, \beta) = \Pi_m(\mathrm{cl}(W^p_{\epsilon(\alpha,\beta)})) = \mathrm{cl}(\Pi_m(W^p_{\epsilon(\alpha,\beta)})) = \mathrm{cl}(\Pi_m(V_{\epsilon(\alpha,\beta)})). \tag{9.1}$$

Let $n(p) := k+1+1+m+m+l'$, and let $x' = (\theta, t, s, y, y', z')$ range over $R^{n(p)}$. Consider also the Λ-set

$$\widetilde{W}^p := \{x' : \ (\theta, y', z') \in V, \ d(y, y') \leq t, \ \text{and } ||z'|| \leq s\}.$$

Before we show that $Y^p(\alpha)$ and Y^p are obtained from $\widetilde{W}^p$, we introduce some more notation. Define the set

$$U := \{x' : \ (\theta, y', z') \in V, \ d(y, y') < t, \ \text{and } ||z'|| < s\},$$

and observe that for each α, β, δ, and ρ the set $\Pi_m(U_{\epsilon(\alpha,\beta),\rho,\delta})$ is open in R^m. Also, for each α, β and ρ such that $Y^p(\alpha, \beta)$ is nonempty, we have

$$S(Y^p(\alpha, \beta), \rho) \subset \bigcup_{\delta \in \mathbb{N}^*} \Pi_m(U_{\epsilon(\alpha,\beta),2\rho,\delta}). \tag{9.2}$$

To see this, choose $y \in S(Y^p(\alpha, \beta), \rho)$. By definition, there is $y'' \in Y^p(\alpha, \beta)$ such that $d(y, y'') \leq \rho$. Thus by (9.1) above, there is $(y', z') \in V_{\epsilon(\alpha,\beta)}$ such that $d(y, y') < 2\rho$.

From Proposition 6.8, we deduce that for each α, β, and ρ there is a γ such that

$$S(Y^p(\alpha, \beta), \rho) \subset \Pi_m(U_{\epsilon(\alpha,\beta),2\rho,\gamma}). \tag{9.3}$$

Unravelling definitions, we also see that for all α, β, δ, and γ,

$$\Pi_m(U_{\epsilon(\alpha,\beta),\rho,\delta}) \subseteq \Pi_m(\widetilde{W}^p_{\epsilon(\alpha,\beta),\rho,\delta}) \subseteq S(\Pi_m V_{\epsilon(\alpha,\beta)}, \rho) \subseteq S(Y^p(\alpha, \beta), \rho). \tag{9.4}$$

We are now set up to prove the claim. First, fix α such that $Y^p(\alpha)$ is nonempty. Using Lemma 6.7, we may assume that $5\rho(\alpha, \beta+1) < \rho(\alpha, \beta)$ for all β in $\mathbb{N}^*$. For each β, define $\gamma(\beta)$ to be the least γ such that

$$S(Y^p(\alpha, \beta), 2\rho(\alpha, \beta)) \subset \Pi_m(U_{\epsilon(\alpha,\beta),4\rho(\alpha,\beta),\gamma(\beta)})$$

as given by (9.3). Then for all β in $\mathbb{N}^*$,

$$
\begin{aligned}
S(Y^p(\alpha), \rho(\alpha,\beta)) &\subseteq S(Y^p(\alpha,\beta), 2\rho(\alpha,\beta)) \\
&\subseteq \Pi_m(\widetilde{W}^p_{\epsilon(\alpha,\beta),4\rho(\alpha,\beta),\gamma(\beta)}) \\
&\subseteq S(Y^p(\alpha,\beta), 4\rho(\alpha,\beta)) \\
&\subseteq S(Y^p(\alpha), 5\rho(\alpha,\beta)).
\end{aligned}
$$

It follows that $Y^p(\alpha)$ is the decreasing intersection of the definable sequence $(\Pi_m(\widetilde{W}^p_{\epsilon(\alpha,\beta),4\rho(\alpha,\beta),\gamma(\beta)}))_\beta$ of closed and bounded subsets of R^m.

Now assume that Y^p is nonempty. Then by Lemma 6.7 we may assume that $8\rho(\alpha+1) < \rho(\alpha)$ for all α. Now for each α, define $\beta(\alpha)$ to be the least β satisfying $\rho(\alpha,\beta) < \rho(\alpha)$. Likewise, define $\gamma(\alpha)$ to be the least γ satisfying

$$S(Y^p(\alpha,\beta(\alpha)), 3\rho(\alpha)) \subset \Pi_m(U_{\epsilon(\alpha,\beta(\alpha)),6\rho(\alpha),\gamma(\alpha)})$$

as given by (9.3). Then for all α in $\mathbb{N}^*$,

$$
\begin{aligned}
S(Y^p, \rho(\alpha)) &\subseteq S(Y^p(\alpha), 2\rho(\alpha)) \\
&\subseteq S(Y^p(\alpha,\beta(\alpha)), 3\rho(\alpha)) \\
&\subset \Pi_m(\widetilde{W}^p_{\epsilon(\alpha,\beta(\alpha)),6\rho(\alpha),\gamma(\alpha)}) \\
&\subseteq S(Y^p(\alpha,\beta(\alpha)), 6\rho(\alpha)) \\
&\subseteq S(Y^p(\alpha), 7\rho(\alpha)) \\
&\subseteq S(Y^p, 8\rho(\alpha)).
\end{aligned}
$$

Once again this means that Y^p is the decreasing intersection of the closed and bounded sets $(\Pi_m(\widetilde{W}^p_{\epsilon(\alpha,\beta(\alpha)),6\rho(\alpha),\gamma(\alpha)}))_\alpha$. This finishes the proof of Claim (1).

Claim (2). Y contains $\mathrm{bd}(X)$.

Fix $a \in \mathrm{bd}(X)$ and choose an arbitrary $t > 0$. Since Y is closed, it suffices to show that $d(a, Y) < 4t$.

Define $d(\alpha) := d(a, W(\alpha))$ for each α in $\mathbb{N}^*$. Since X is the union of the increasing sequence $(W(\alpha))_\alpha$, it follows that $\lim_{\alpha\to\infty} d(\alpha) = 0$. Thus there is an

α in I so large that $\rho_p(\alpha) < t$ for all p in S and $d(\alpha) < t$. Fix such an α, and choose b in $W(\alpha)$ such that $d(a,b) = d(\alpha)$. Such a b exists since $W(\alpha)$ is closed.

Since it follows that $b \in \mathrm{bd}(W(\alpha))$, the set $\mathcal{B} := B(b,t)\backslash W(\alpha)$ is nonempty. Choose a point $b' \in \mathcal{B}$. Since the set $W(\alpha)$ is the decreasing intersection of the sets $W(\alpha,\beta)$, we conclude that $b' \notin W(\alpha,\beta)$ for all sufficiently large β. Thus we may choose a β in J so large that $b' \notin W(\alpha,\beta)$ and $\rho_p(\alpha,\beta) < t$ for all p in S.

Now consider the line segment $[b,b']$ joining b and b' in $\mathbb{R}^n$. Since the set $W(\alpha,\beta) \cap [b,b']$ is closed and nonempty, it must contain a point c closest to b'. Note that $d(a,c) < 2t$. But there also must be a p in S such that c is in $\Pi_m(W^p_{\epsilon(\alpha,\beta)})$; this is because the set $\Pi_m(W^p_{\epsilon(\alpha,\beta)})$ is open in R^n whenever p is not in S by the Rank Theorem (Corollary 2.14). Since then c is in $Y^p(\alpha,\beta)$, we also have $d(c, Y^p) < 2t$. This finishes the proof of Claim (2).

Claim (3). Y has empty interior.

We prove the following by induction on m: Let a subset X of R^m be a Λ^∞-set obtained from a Λ-set W. Let W be the union of Λ-sets $W^1, \dots, W^K$ that are $\mathcal{R}$-manifolds in standard position. Let Y be the (I,J)-approximation of $\mathrm{bd}(X)$. Then Y has empty interior.

If $m = 1$, then $d_p = 0$ for each p in S. Then by the Rank Theorem, for each ϵ in R^l and each point x in W^p_ϵ, there is a rational box U containing x such that the projection $\Pi_1(U \cap W^p_\epsilon)$ is a singleton $\{r\}$. It follows from the Baire Category Theorem that the set $\Pi_1(W^p_\epsilon)$ has empty interior. From Axiom (VI) for Λ-sets, we know that the number of components in $W^p_{\epsilon(\alpha,\beta)}$ is uniformly bounded. Consequently, the set Y^p is finite for each p in S.

Now let $m > 1$, and assume that the result holds for lower values of m. Consider the set

$$H := \{a \in R : Y_a \text{ is not the } (I,J)\text{-approximation of } \mathrm{bd}(X_a)\}.$$

Since X_a is also obtained from W, the set Y_a has empty interior for each $a \in R\backslash H$ by the inductive hypothesis. By Lemma 8.4(ii), it suffices to show that H has empty interior.

Now since each W^p is in standard position, each fiber $W^p_{\epsilon,a}$ is an $\mathcal{R}$-manifold in standard position (Corollary 3.7). Moreover, there is an $e_p \leq m-1$ independent of (ϵ, a) such that the projection $\Pi_{m-1}|_{W^p_{\epsilon,a}}$ has constant rank e_p. We define

$$S' := \{p \in \{1, \ldots, K\} : e_p < m-1\}.$$

Then by definition, the set H is contained in the union of the following sets:

$$\begin{aligned}
G^p(\alpha,\beta) &:= \{a \in R : \Pi_{m-1}(\mathrm{cl}(W^p_{\epsilon(\alpha,\beta),a})) \neq Y^p(\alpha,\beta)_a\},\\
G^p(\alpha) &:= \{a \in R : Y^p(\alpha)_a \neq \emptyset \text{ and } \lim(Y^p(\alpha,\beta)_a)_{\beta\in J} \neq Y^p(\alpha)_a.\},\\
G^p &:= \{a \in R : Y^p_a \neq \emptyset \text{ and } \lim(Y^p(\alpha)_a)_{\alpha\in I} \neq Y^p_a\},\\
G &:= \{a \in R : \textstyle\bigcup_{p\in S} Y^p_a \neq \bigcup_{p\in S'} Y^p_a\}.
\end{aligned}$$

By Proposition 6.6, it suffices to show that these sets are $\mathcal{R}$-countable for α in I and β in J.

For each p, we have $G^p(\alpha,\beta) \subseteq \{a \in R : \mathrm{cl}(W^p_{\epsilon(\alpha,\beta),a}) \neq \mathrm{cl}(W^p_{\epsilon(\alpha,\beta)})_a\}$, so each $G^p(\alpha,\beta)$ is $\mathcal{R}$-countable by Lemma 9.1. By Lemma 9.2 and Claim (1), the sets G^p and $G^p(\alpha)$ are $\mathcal{R}$-countable for all α in I. It remains to show that G is $\mathcal{R}$-countable.

Let

$$\mathcal{G}^p := G \backslash \left(G^p \cup \bigcup_{\alpha\in I} \left(G^p(\alpha) \cup \bigcup_{\beta\in J} G^p(\alpha,\beta) \right) \right).$$

It suffices to show that $\mathcal{G}^p$ is $\mathcal{R}$-countable. Notice that S' is a subset of S, so that if a is in G, there must be a $p \in S \backslash S'$ such that Y^p_a is not empty. Fix such a p. Note that if a is in $\mathcal{G}^p$, then there are α and β in $\mathbb{N}^*$ such that $W^p_{\epsilon(\alpha,\beta),a}$ is not empty. Thus it suffices to show that $\{a \in R : W^p_{\epsilon(\alpha,\beta),a} \neq \emptyset\}$ is $\mathcal{R}$-countable. We in fact show that it is finite.

Fix α in I, β in J, and to simplify notation set $\epsilon := \epsilon(\alpha,\beta)$. Notice now that $d_p = e_p = m-1$ and that the dimensions of W^p_ϵ and $W^p_{\epsilon,a}$ are the same whenever the fiber $W^p_{\epsilon,a}$ is not empty: Given $b \in \Pi_{m-1}(W^p_{\epsilon,a})$, we have the equalities

$$\dim(W^p_\epsilon) - d_p = \dim(W^p_{\epsilon,a,b}) = \dim(W^p_{\epsilon,a}) - e_p,$$

by Theorem 3.4. Consequently, the set $\{a\} \times W^p_{\epsilon,a}$ is relatively open (and closed) in W^p_ϵ for each a in R. Hence the number of nonempty fibers $W^p_{\epsilon,a}$ cannot exceed the number of components of W^p_ϵ. Since this number is finite by Axiom (VI), Claim (3) is proved. This completes the proof of Proposition 8.5. $\square$

And that completes the proof of Theorem 1.8.

10 T-absoluteness

Now that we have generalized Theorem 1.6 to arbitrary models of T^{proj}, we need to study how to pass information from one model to another. In particular, we want to use model-theoretic compactness to prove Theorem 1.10—but in order for the argument to work, we need to know something like the following: Suppose a formula ϕ defines an $\mathcal{S}$-Rolle leaf in a model $\mathcal{S}$ of T^{proj}. Then ϕ also defines an $\mathcal{R}$-Rolle leaf in a model $\mathcal{R}$ of T^{proj}. We clarify this after resetting our assumptions.

We relax for the moment our assumption that $\mathcal{R}$ models T^{proj}. Instead, we require only that $\mathcal{R}$ has the IVP. We let T denote the theory and $\mathscr{L}$ the language of $\mathcal{R}$. All formulas below are assumed to be $\mathscr{L}$-formulas. The structure $\mathcal{S} = \langle S, \dots \rangle$ here denotes another model of T, and for convenience we assume that $\mathcal{R}$ and $\mathcal{S}$ are both elementary submodels of a common model of T.

Here are the main questions: Let ϕ be a formula, and let $\mathcal{P}$ be some topological property of sets. Suppose that the interpretation $\phi(R^n)$ of ϕ in R^n has property $\mathcal{P}$. Does it follow that the interpretation $\phi(S^n)$ of ϕ in S^n also has property $\mathcal{P}$? Suppose, moreover, that we have a definable family $\{\phi(R^n, b) : b \in R^m\}$. Is the set

$$\{b \in R^m : \phi(R^n, b) \text{ has property } \mathcal{P}\}$$

definable? Does the same formula define the analogous set in $\mathcal{S}$?

In case that $\mathcal{P}$ is the property of being open, the answer to all of these questions is easily seen to be yes: let $d_\phi(y)$ be a formula expressing that there is, for each point x in $\phi(R^n, y)$, an open box B around x contained in X. Then for all c in S, we have that $\mathcal{S} \models d_\phi(c)$ if and only if the set $\phi(S^n, c)$ is open. Indeed, this fact has already been used in Chapter 6. This case is the ideal situation and the motivation for the next definition.

Definition 10.1. Suppose $\mathcal{P}$ is a property of sets. We say $\mathcal{P}$ is T**-absolute** if for each formula $\phi(x, y)$ there is a formula $d_\phi(y)$ such that for each model $\mathcal{S}$ of T and all c in S^m we have the following:

The set $\phi(S^n, c)$ has property $\mathcal{P}$ if and only if $\mathcal{S} \models d_\phi(c)$.

As discussed above, the property of being open (in the ambient universe) is T-absolute. It follows that the property of being closed is also T-absolute. It is also elementary to prove that the following properties are T-absolute:

- $\mathcal{P}(X) :=$"X is the graph of a definable function."
- $\mathcal{P}(X) :=$"X is the graph of a definable injection."
- $\mathcal{P}(X) :=$"X is the graph of a function of class C^k (for fixed k)."
- $\mathcal{P}(X) :=$"X is the graph of a definable path $\gamma : [0, 1] \to R^n$ of class C^1."
- $\mathcal{P}(X) :=$"X is the graph of a nonvanishing vector field of class C^k (for fixed k)."

Now what happens if the property $\mathcal{P}$ is that of being definably connected? When is definable connectedness T-absolute? Here is a first approximation to what we want:

Proposition 10.2. *Let $\phi(x, y)$ be a formula, and let K be a natural number such that for all b in R^m the set $\phi(R^n, b)$ has fewer than K components. Then for any model $\mathcal{S} = \langle S, <, \dots \rangle$ of T and any b' in S, the set $\phi(S^n, b')$ has fewer than K components.*

Proof. Let $\mathcal{S}$ be a model of T, and suppose for a contradiction that the set $X_{b'} := \phi(S^n, b')$ has at least K components for some b' in S^m. Then there are K $\emptyset$-definable families $U^1_z, U^2_z, \dots U^K_z$ of subsets of S^n and a c in S^l such that

(1) each U^i_c is an open subset of S^n,

(2) the sets U_c^i are pairwise disjoint,

(3) $X_{b'} \subseteq \bigcup_{i=1,\dots,K} U_c^i$, and

(4) for each $i = 1, \dots, K$ the set $U_c^i \cap X_{b'}$ is not empty.

One can write down a first-order parameter-free formula that expresses that there are b' and c satisfying all of these conditions. This formula must hold in any model of T, a contradiction. □

For full T-absoluteness of definable connectedness, we bump up against an equivalence.

Proposition 10.3. *The following are equivalent:*

(1) *Definable connectedness is T-absolute.*

(2) *For each formula $\phi(x, y)$ there is a formula $\psi_\phi(x, y)$ such that for all c in R^m, the set $\phi(R^n, c)$ is not definably connected if and only if the set $\psi_\phi(R^n, c)$ is a proper nonempty closed and open subset of $\phi(R^n, c)$.*

Proof. For the direction (1)⇒(2) suppose that definable connectedness is T-absolute. Let $\phi(x, y)$ be a formula, and suppose for a contradiction that there is no corresponding $\psi_\phi(x, y)$. Let the formula $d_\phi(y)$ be as in the definition of T-absoluteness, and let $\chi_\psi(y)$ be a formula expressing that $\psi(R^n, y)$ is a nonempty proper closed and open subset of $\phi(R^n, y)$. Then the type

$$\Phi := \{\neg\chi_\psi(y) : \psi \text{ an } \mathscr{L}_{\text{proj}} - \text{formula}\} \cup \{\neg d_\phi(y)\}$$

is consistent. Consequently, this type is realized in some model of T by the compactness theorem, a contradiction.

For the converse, simply take $d_\phi(y)$ to be $\neg\chi_\psi(y)$. □

Under the assumption of ω-saturation, here is a related fact:

Proposition 10.4. *Suppose $\mathcal{R}$ is ω-saturated and that definable connectedness is T-absolute. Suppose that X_y is a definable family such that each fiber X_b has only finitely many components. Then there is a natural number K such that each fiber has fewer than K components.*

Proof. Using Proposition 10.3, there are families X^0_y and X^1_y such that whenever X_b is not definably connected, X^0_b and X^1_b partition X_b into two relatively closed and open sets. Assuming that members of the family X_y have arbitrarily large numbers of components, the same must be true of one of X^0_y or X^1_y. Repeating this argument, we construct a sequence $\sigma \in 2^\omega$ such that for every finite truncation $\widehat{\sigma}$ of σ, the members of the family $X^{\widehat{\sigma}}_y$ have arbitrarily large numbers of components. In this way we obtain a consistent type

$$p(y) := \{X^{(\widehat{\sigma},0)}_y \neq \emptyset \wedge X^{(\widehat{\sigma},1)}_y \neq \emptyset : \widehat{\sigma} \text{ is a finite truncation of } \sigma\}.$$

Choose a b that realizes p; then X_b has infinitely many components. □

We now turn our attention back to T^{proj}. From now on we assume that $\mathcal{R}$ is a model of T^{proj}. We note first that definable connectedness *is* T^{proj}-absolute.

Lemma 10.5. *Let K be a natural number. The following properties $\mathcal{P}$ are T^{proj}-absolute:*

- $\mathcal{P}(X) :=$ *"X has at least K components."*
- $\mathcal{P}(X) :=$ *"X has at least K p-components.*

Proof. To prove the first statement, let X^n be the definable subset of R^{n+1} given by Corollary 6.2, the fibers of which are precisely the open subsets of R^n. For a formula $\phi(x, y)$, let $d_\phi(y)$ be a formula expressing that there is a K-tuple $(r_1, r_2, \ldots, r_K)$ in R^K such that the sets $X^n_{r_i}$ are pairwise disjoint, each has nonempty intersection with the set $\phi(R^n, y)$, and their union covers $\phi(R^n, y)$.

For the second statement, let $d_\phi(y)$ be a formula that expresses that for each $i = 1, \ldots, K$, there is an a_i in $\phi(R^n, y)$ such that whenever $1 \leq i < j \leq K$, there is no definable continuous path connecting a_i to a_j whose image lies in $\phi(R^n, y)$. Again, Corollary 6.2 guarantees that such a formula exists. □

Now we check T^{proj}-absoluteness up to the Rolle property.

Lemma 10.6. *Fix natural numbers m and k. Then the following property $\mathcal{P}$ is T^{proj}-absolute:*

- $\mathcal{P}(X) :=$ *"X is a T^{proj}-manifold of dimension m and class C^k."*

Proof. Using Corollary 6.2, there is a first-order formula $d_\phi(y)$ expressing the following:

"For every x in $\phi(R^n, y)$, there is a chart φ for $\phi(R^n, y)$ at x."

Then $d_\phi(y)$ satisfies our requirements. □

Similarly using Lemmas 10.5 and 10.6, we obtain another Lemma.

Lemma 10.7. *Let $(\omega_z)_{z\in R}$ be a parameter-free definable family of 1-forms on a definable family $(U_z)_{z\in R}$ of open sets. Then the following properties are T^{proj}-absolute:*

- $\mathcal{P}(X) :=$ *"There is a parameter c such that X is an integral T^{proj}-manifold of $\omega_c = 0$."*
- $\mathcal{P}(X) :=$ *"There is a parameter c such that X is a Rolle T^{proj}-leaf of $\omega_c = 0$."*

Proof. By Corollary 6.2 and Corollary 10.5, we can definably quantify over all definably path connected $\mathcal{R}$-manifolds of dimension 1 and class C^1. □

We conclude this Chapter by proving Theorem 1.10 from Theorem 1.8. In fact, we prove a slightly more general version after modestly extending the definitions.

Let $\widetilde{\mathscr{L}}$ be the language of some o-minimal structure $\widetilde{\mathcal{S}}$ that expands a real closed field and has an expansion to a model $\mathcal{S}$ of T^{proj}. Let $P_1, \ldots, P_j$ be predicate symbols each of which is neither in $\mathscr{L}_{\text{proj}}$ nor in $\widetilde{\mathscr{L}}$. Let $\Phi = (\phi_0, \ldots, \phi_j)$ be a tuple of $\widetilde{\mathscr{L}}$-formulas.

Definition 10.8. We say that a subset X of S^n **has format** Φ if there are $\mathscr{L}_{\text{proj}}$-formulas χ_i for $i = 1, \ldots, j$ such that the following hold:

(1) Each ϕ_i is in the language $\widetilde{\mathscr{L}} \cup \{P_1, \ldots, P_i\}$, and each χ_i is an $\mathscr{L}_{\text{proj}}$-formula.

(2) For $i = 0, \ldots, j-1$, each $\phi_i[\chi_1, \ldots, \chi_i]$ defines (in $\mathcal{S}$) the graph of a nonvanishing vector field on an open set U_i.

(3) For $i = 0, \ldots, j-1$, each χ_{i+1} defines an $\mathcal{S}$-Rolle leaf of the nonsingular 1-form determined by $\phi_i[\chi_1, \ldots, \chi_i]$.

(4) The set X is defined by the formula $\phi_j[\chi_1, \ldots, \chi_j]$.

Here is the generalization of Theorem 1.10 that we prove:

Theorem 10.9. *There is a natural number K such that whenever*

(i) $\widetilde{\mathcal{S}}'$ *is a structure that is elementarily equivalent to* $\widetilde{\mathcal{S}}$*,*

(ii) $\widetilde{\mathcal{S}}'$ *has an expansion* $\mathcal{S}'$ *to a model of* T^{proj}*,*

(iii) X *is a set that is definable in* $\mathscr{P}(\widetilde{\mathcal{S}}', \mathcal{S}')$*, and*

(iv) X *has format* Φ*,*

then the set X has fewer than K components (with respect to $\mathcal{S}'$).

In other words, the bound K does not even depend on the structure $\widetilde{\mathcal{S}}$.

Proof. For the sake of notation, we assume that each χ_i has n_i free variables. Then we use the variables $x^i := (x_1, ..., x_{n_i})$.

Let $\mathcal{R}$ be an ω-saturated elementary extension of $\mathcal{S}$, and assume first that for each K there is an $\mathcal{R}$-definable set Y^K with format Φ and at least K components. By Proposition 6.2 and Lemma 10.5, for each $i = 1, \ldots, j-1$ there is a formula

$\chi_i(x^i, t)$ such that the set $\{\chi_i(R^{n_i}, r) : r \in R\}$ is equal to the collection of all $\mathcal{R}$-Rolle leaves of all 1-forms corresponding to the definable vector fields

$$\phi_i[\chi_1(x^1, c^1), \ldots, \chi_{i-1}(x^{i-1}, c^{i-1})](x^i) \tag{10.1}$$

as the parameters $c^1, \ldots, c^{i-1}$ vary. By Proposition 10.4, there are in fact parameters $c^1, \ldots, c^{j-1}$ such that the formula

$$\phi_j[\chi_1(x^1, c^1), \ldots, \chi_{j-1}(x^{j-1}, c^{j-1})](x^j) \tag{10.2}$$

defines a set Y with infinitely many components and also such that $\chi_i(x^i, c^i)$ defines a Rolle $\mathcal{R}$-leaf of a 1-form determined by (10.1) for $i = 1, \ldots, j-1$. Hence Y is definable in the o-minimal structure $\mathscr{P}(\mathcal{R}|_{\widetilde{\mathscr{L}}}, \mathcal{R})$. This contradiction shows that there is a K such that every $\mathcal{R}$-definable set with format Φ has fewer than K components.

For the general case, notice that every $\mathcal{S}'$-definable set with format Φ is defined by formula (10.2) for some choice of parameters $c^1, \ldots, c^{j-1}$. The result now follows from Proposition 10.2. □

11 Discussion, connections, future directions

Recall that $\mathcal{R}$ is an elementary extension of $\mathbb{R}_{\text{proj}}$ and that $\widetilde{\mathscr{L}}$ is the language of an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$. In the proof of Theorem 1.10, we used the structure $\mathscr{P}(\mathcal{R}|_{\widetilde{\mathscr{L}}}, \mathcal{R})$ to obtain information about the standard Pfaffian closure $\mathscr{P}(\widetilde{\mathbb{R}})$. Now that we have proved that the structure $\mathscr{P}(\mathcal{R}|_{\widetilde{\mathscr{L}}}, \mathcal{R})$ is o-minimal, other questions quickly emerge.

For instance, is the structure $\mathscr{P}(\mathcal{R}|_{\widetilde{\mathscr{L}}}, \mathcal{R})$ interdefinable with the reduct of $\mathcal{R}$ to the language $\mathscr{L}_{\text{Pfaff}}$ of the structure $\mathscr{P}(\widetilde{\mathbb{R}})$? If this were the case, then we could have easily obtained the o-minimality of $\mathscr{P}(\mathcal{R}|_{\widetilde{\mathscr{L}}}, \mathcal{R})$ through elementary arguments. At present, we cannot rule out this possibility; however, we take some consolation in the fact that were this the case it would also imply a positive answer to Hilbert's 16th problem. This Chapter explores such connections and ties up some loose ends.

Consider the following statement about $\mathscr{P}(\widetilde{\mathbb{R}})$:

Property 11.1. *(cf. Lemma 10.5) For each n and each $\mathscr{P}(\widetilde{\mathbb{R}})$-definable family of 1-forms $(\omega_z)_{z\in R^k}$ there is a $\mathscr{P}(\widetilde{\mathbb{R}})$-definable set $X \subset \mathbb{R}^{m+n}$ such that*

$$\{L \subset \mathbb{R}^n : L \text{ is a Rolle leaf of } \omega_c = 0 \text{ for some } c \in R^k\} \subseteq \{X_a : a \in \mathbb{R}^m\}.$$

We shall express Property 11.1 with the phrase: "Rolle leaves are uniformly definable in $\mathscr{P}(\widetilde{\mathbb{R}})$."

Proposition 11.2. *The following are equivalent:*

(1) *Rolle leaves are uniformly definable in $\mathscr{P}(\widetilde{\mathbb{R}})$.*

(2) *Whenever $\mathcal{R}$ is an elementary extension of $\mathbb{R}_{\text{proj}}$, and Y is definable in $\mathscr{P}(\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}, \mathcal{R})$, then Y is already definable in $\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}$.*

Proof. Assume (1) holds. By the construction of $\mathscr{P}(\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}, \mathcal{R})$, it suffices for (1⇒2) to show (2) under the assumption that Y is a Rolle leaf of $\omega = 0$ for some $R_{\mathscr{L}_{\text{Pfaff}}}$-definable 1-form ω. Any such ω is a member of a definable family $(\omega_z)_{z\in R^k}$ of $\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}$-definable 1-forms, and T^{proj} implies the following statement: "if L is a Rolle $\mathcal{R}$-leaf of $\omega_c = 0$ for some c, then there is an a in R^m such that $L = X_a$." Thus Y is $\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}$-definable.

Conversely suppose (1) fails, and let $(\omega_z)_{z\in R^k}$ be a definable family of 1-forms that witnesses this. By Corollary 6.2 and Lemma 10.5, there is an $\mathbb{R}_{\text{proj}}$-definable family (Z_y) such that

$$\{Z_b : b \in R^m\} = \{L \subset \mathbb{R}^n : L \text{ is a Rolle leaf of } F_c \text{ for some } c \in R^k\}.$$

But then the type

$$p(y) := \{\forall w(Z_y \neq W_w) : W_w \text{ is a } \mathscr{P}(\widetilde{\mathbb{R}})\text{-definable family}\}$$

is consistent. Hence $p(y)$ is realized by some b in some elementary extension $\mathcal{R}$ of $\mathbb{R}_{\text{proj}}$. Again by Lemma 10.5, the set Z_b is a Rolle $\mathcal{R}$-leaf, but Z_b is not $\mathcal{R}|_{\mathscr{L}_{\text{Pfaff}}}$-definable. □

We next describe a proof of the fact that if Rolle leaves are uniformly definable in the structure $\mathscr{P}(\widetilde{\mathbb{R}})$, then we obtain a positive answer to Hilbert's 16th problem. Without providing details, we use the set-up of Dolich and Speissegger in [3], which itself is inspired by [21].

Recall that, given a (possibly singular) 1-form ω on $\mathbb{R}^2$, a **limit cycle** of $\omega = 0$ is a compact leaf L of $\omega = 0$ for which there is a noncompact leaf L' such that $L \subset \text{cl}(L')$. **Hilbert's 16th problem** asks for a proof of the following statement:

> There is a function $H : \mathbb{N} \to \mathbb{N}$ such that for all d in $\mathbb{N}$, if $\omega = p(x_1, x_2)dx_1 + q(x_1, x_2)dx_2$ is a 1-form defined on $\mathbb{R}^2$ with polynomial coefficients p and q each of degree at most d, then ω has at most $H(d)$ limit cycles.

For any 1-form ω on R^2 with analytic coefficients, a construction in [3] yields a function $\mathfrak{f} : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$, called a **progression map** associated to ω, with the following properties:

(i) There is a 1-dimensional definable subset B of $\mathbb{R}^2$ such that B intersects each compact cycle, and the image of $\mathfrak{f}$ is contained in $B \cup \{\infty\}$.

(ii) There is a natural number l such that for every x in B the point x is in the set $\mathrm{Fix}_B(\mathfrak{f}^l) := \{x \in B : \mathfrak{f}^l(x) = x\}$ if and only if there is a compact leaf of $\omega = 0$ that contains x. (Here $\mathfrak{f}^l$ denotes the l-th iterate of $\mathfrak{f}$.)

Notice that each limit cycle contains a point of $\mathrm{bd}(\mathrm{Fix}_b(\mathfrak{f}^l))$. Consequently, if the set $\Gamma(\mathfrak{f}) \cap \mathbb{R}^4$ is definable in the o-minimal structure $\mathscr{P}(\widetilde{\mathbb{R}})$, then the set $\mathrm{bd}(\mathrm{Fix}_B(\mathfrak{f}^l))$ is definable and 0-dimensional—that is to say, finite.

Assume now that Rolle leaves are uniformly definable in $\mathscr{P}(\widetilde{\mathbb{R}})$. Then the construction in [3] can be done definably and uniformly for families of 1-forms definable in $\mathscr{P}(\widetilde{\mathbb{R}})$. In particular, for the family (ω_z) of all polynomial 1-forms of degree at most d, there are definable families $(\mathfrak{f}_z)$ and (B_z) such that for each choice of parameters c, the function $\mathfrak{f}_c$ and the set B_c correspond to ω_c as described above. Moreover, the number l can be chosen independently of the parameter c. Since the family $\mathrm{bd}(\mathrm{Fix}_{B_z}(\mathfrak{f}_z^l))$ is definable and uniformly finite, the number of limit cycles of ω_z is bounded.

Remarks: Hilbert's 16th problem remains open, but a variant known as Dulac's problem has been settled. In the case that ω has analytic coefficients, it was shown independently by Il'yashenko [11] and Ecalle et al. [5],[6] that $\omega = 0$ has only finitely many limit cycles. These proofs corrected an erroneous proof given by Dulac in [4].

We next explore some other issues raised by Theorem 1.8. Suppose we are given an o-minimal expansion $\widetilde{\mathcal{R}}$ of a field $\overline{\mathcal{R}}$. If we wish to find an o-minimal

Pfaffian closure as above, we need to produce a set $\mathbb{Z}^*$ such that $\langle \overline{\mathcal{R}}, \mathbb{Z}^* \rangle$ expands $\widetilde{\mathcal{R}}$ and models T^{proj}. Rephrased as a question,

Question 1. *When is an o-minimal expansion of a real closed field compatible with the projective hierarchy?*

Even in the event that we can find such an expansion, to what extent is the induced Pfaffian closure unique? That is,

Question 2. *Suppose $\mathcal{R}$ and $\mathcal{R}'$ are two expansions of $\widetilde{\mathcal{R}}$ that model T^{proj}. Is the structure $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ isomorphic to $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R}')$?*

Turning to what is known, we offer two settings where Theorem 1.8 does apply.

Proposition 11.3. *Suppose $\mathcal{R}$ is a countable elementary substructure of $\mathbb{R}_{\text{proj}}$, and $\widetilde{\mathcal{R}}$ is any o-minimal reduct of $\mathcal{R}$ with a countable language. Then $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ has a countable language.*

Remarks: In this case, the structure $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ is an o-minimal Pfaffian closure that is suitable for the questions of computable model theory. It also defines an exponential function since $\mathbb{R}_{\text{proj}}$ does. It is conceivable that such structures may be related to Tarski's question of the decidability of $\langle \overline{\mathbb{R}}, \mathbb{Z} \rangle$. (See [26].)

Proof. It suffices to prove the proposition for $\mathscr{P}_1(\widetilde{\mathcal{R}}, \mathcal{R})$. For each n there are only countably many vector fields that are definable in $\widetilde{\mathcal{R}}$ on an open subset of R^n since the language and universe are countable. Moreover, for each vector field, there is at most one Rolle $\mathcal{R}$-leaf through any given point by Lemma 5.4. Thus $\mathscr{P}_1(\widetilde{\mathcal{R}}, \mathcal{R})$ is an expansion of $\widetilde{\mathcal{R}}$ by countably many predicates. □

On the other end of the spectrum, we can take Pfaffian closures of arbitrarily saturated models.

Proposition 11.4. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of $\overline{\mathbb{R}}$. Then for every cardinal κ, there is a κ-saturated elementary extension $\widetilde{\mathcal{R}}$ of $\widetilde{\mathbb{R}}$ and an expansion*

$\mathcal{R}$ of $\widetilde{\mathcal{R}}$ that models $\mathbb{R}_{\text{proj}}$.

Proof. This is a corollary of the existence of what Hodges calls κ-big models. (See [10].) □

We turn at last to the question that inspired this research. Is the IVP enough? Let us restate it as a conjecture:

Conjecture 11.5. *Let $\widetilde{\mathcal{R}}$ be an o-minimal expansion of a real closed field and let $\mathcal{R}$ be an expansion of $\widetilde{\mathcal{R}}$ with the* IVP. *Then $\mathscr{P}(\widetilde{\mathcal{R}}, \mathcal{R})$ is o-minimal.*

In actuality, what we are after is slightly stronger. We would like this conjecture to be true with the Alternate $\mathcal{R}$-Rolle Property of Chapter 5 in place of the $\mathcal{R}$-Rolle Property. Why is this? Here is one reason:

Corollary 11.6 (of 11.5 with the Alternate $\mathcal{R}$-Rolle Property). *Let $\mathcal{R}_E := \langle \overline{\mathcal{R}}, E \rangle$ be an expansion of the real closed field $\overline{\mathcal{R}}$ by a unary function $E : R \to R$. Suppose that $\mathcal{R}_E$ has the* IVP *and that E is a function of class C^1 satisfying the differential equation $E' = E$. Then $\mathcal{R}_E$ is o-minimal.*

Proof. We need only show that E has the Alternate $\mathcal{R}_E$-Rolle Property with respect to the 1-form $\omega := ydx - dy$. This is proved exactly as in example 1.3 of [25]. □

While the IVP is not much easier to check than o-minimality, it is easier to axiomatize. In other words, assuming the strong version of Conjecture 11.5, we can explicitly write down an (incomplete) axiom scheme Φ for $\mathcal{R}_E$ such that every model of Φ is o-minimal. Namely, in addition to an axiom describing the field structure, and one expressing that $E' = E$, we add an axiom for each formula in one free variable $\phi(x)$ that expresses:

If $\phi(x)$ defines a nonempty closed and open subset of R,
then $\phi(x)$ defines R.

Finally, Conjecture 11.5 with the Alternate $\mathcal{R}$-Rolle Property seems to require an analogous version of Theorem 1.7. To prove this, it suffices to know that a definably connected $\mathcal{R}$-manifold is definably path connected. At present, this is not even known in the case where $\mathcal{R}$ is a reduct of $\mathbb{R}_{\text{proj}}$.

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